On the minimization properties of the Tisserand systems

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1. Introduction

When a collection of material particles $\mathcal{S}$ experience relative displacements, it is no possible to define unambiguously a rotational motion of the set. In these situations it is assigned to $\mathcal{S}$ a reference system $Oxyz$ (the "body axes") with origin in its barycenter O and connected with it in a prescribed way.

By doing so, the rotation of the particles is identified with the rotation of the body axes with respect to some inertial, or quasi–inertial, reference system $OXYZ$. This rotation admits a precise definition in mathematical terms.

There are different possibilities to connect the body axes $Oxyz$ with the considered set of particles (Munk & McDonald 1960). From the point of view of simplifying the equations of motion, one convenient method is to employ the so–called Tisserand systems (Tisserand 1891).

2. Tisserand systems

To introduce Tisserand systems, let us write the velocity, relative to $OXYZ$, of a particle of $\mathcal{S}$ with position $\vec{x}_i$ and mass m_i as

$$\vec{V}_i = \vec{\omega} \times \vec{x}_i + \vec{v}_i(\vec{\omega}). \tag{1}$$

The vector $\vec{\omega}$ is, at this stage, arbitrary and common for the set $\mathcal{S}$. In contrast $\vec{v}_i(\vec{\omega})$, the deformation velocity (Moritz & Mueller 1987), depends on the material particle i and the particular choice of $\vec{\omega}$.

Tisserand systems can be defined by any of the following conditions that fix $\vec{\omega}$ to a particular value $\vec{\omega}_T$:

(a) The angular momentum of $\mathcal{S}$

$$\vec{L} = \sum_{i\in\mathcal{S}} m_i \left(\vec{x}_i \times \vec{V}_i\right) \tag{2}$$

is $\vec{L} = \mathsf{I}\,\vec{\omega}_T$ (Tisserand 1891), where I is the matrix of inertia of $\mathcal{S}$

(b) The kinetic energy of $\mathcal{S}$ associated to the deformation velocity

$$\mathcal{T}_{\rm def}(\vec{\omega}_T) = \frac{1}{2}\sum_{i\in\mathcal{S}} m_i \vec{v}_i(\vec{\omega}_T)^2 \tag{3}$$

is minimum (Jeffreys 1976)

(c) The angular momentum of $\mathcal{S}$ related with the deformation velocity

$$\vec{h}(\vec{\omega}_T) = \sum_{i\in\mathcal{S}} m_i \left[\vec{x}_i \times \vec{v}_i(\vec{\omega}_T)\right] \tag{4}$$

is the null vector (Tisserand 1891)

3. Equivalence of the conditions

The former characterizations turn out to be equivalent, that is to say, (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a). The second and third implications are detailed in the existing literature (e.g., Moritz & Mueller 1987). Let us focus on the first one.

From Eqs. (1) and (3), the deformation kinetic energy can be written as (Escapa 2011)

$$\mathcal{T}_{\rm def}(\vec{\omega}) = \mathcal{T} - \vec{L}\,\vec{\omega} + \frac{1}{2}\vec{\omega}\,\mathsf{I}\,\vec{\omega}, \tag{5}$$

where $\mathcal{T}$ is the kinetic energy of $\mathcal{S}$. Hence, for an arbitrary vector $\vec{\lambda}$ different from $\vec{0}$, we have

$$\mathcal{T}_{\rm def}\left(\vec{\omega}+\vec{\lambda}\right) = \mathcal{T}_{\rm def}(\vec{\omega}) - \vec{L}\,\vec{\lambda} + \vec{\lambda}\,\mathsf{I}\,\vec{\omega} + \frac{1}{2}\vec{\lambda}\,\mathsf{I}\,\vec{\lambda}. \tag{6}$$

If we consider condition (a), defining the angular momentum of the system $\vec{L}$, in Eq. (6), we get

$$\mathcal{T}_{\rm def}\left(\vec{\omega}_T+\vec{\lambda}\right) - \mathcal{T}_{\rm def}(\vec{\omega}_T) = \frac{1}{2}\vec{\lambda}\,\mathsf{I}\,\vec{\lambda}. \tag{7}$$

Since the matrix of inertia is definite positive, we have that

$$\frac{1}{2}\vec{\lambda}\,\mathsf{I}\,\vec{\lambda} > 0,\ \vec{\lambda}\in\mathbb{R}^3,\ \vec{\lambda}\neq\vec{0}. \tag{8}$$

Therefore, Eq. (7) implies that $\mathcal{T}_{\rm def}(\vec{\omega})$ takes its minimum at $\vec{\omega}_T$, i.e., condition (b).

4. Tisserand systems evolution

The angular velocity $\vec{\omega}_T$, considered as a known function of time, determines the rotational kinematics of the body axes, but not its orientation in an univocal manner (Tisserand 1891).

Specifically, from the components of $\vec{\omega}_T$ in the $OXYZ$ system, we can construct the skew–symmetric matrix

$$\Sigma_T(t) = \begin{pmatrix} 0 & -\omega_{TZ}(t) & \omega_{TY}(t) \\ \omega_{TZ}(t) & 0 & -\omega_{TX}(t) \\ -\omega_{TY}(t) & \omega_{TX}(t) & 0 \end{pmatrix}. \tag{9}$$

It allows defining a rotation matrix $\mathsf{R}(t)$ that brings the $OXYZ$ system to the body axes through (Wintner 1941)

$$\Sigma_T(t) = \frac{d\mathsf{R}^T}{dt}\mathsf{R}, \tag{10}$$

where the superscript T denotes the transpose of a matrix. The solution of this linear differential equation is given by

$$\mathsf{R}(t) = \mathsf{R}(t_0)\exp\left(-\int_{t_0}^{t}\Sigma_T(s)ds\right), \tag{11}$$

$\mathsf{R}(t_0)$ providing the numerical value of $\mathsf{R}(t)$ at the epoch t_0.

In this way, besides any of the conditions (a), (b), or (c), the specification of a particular Tisserand system requires providing explicitly the initial orientation of the body axes relative to $OXYZ$.

References

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