Do we need various assumptions to get a good FCN?

-- A new multiple layer spectral method

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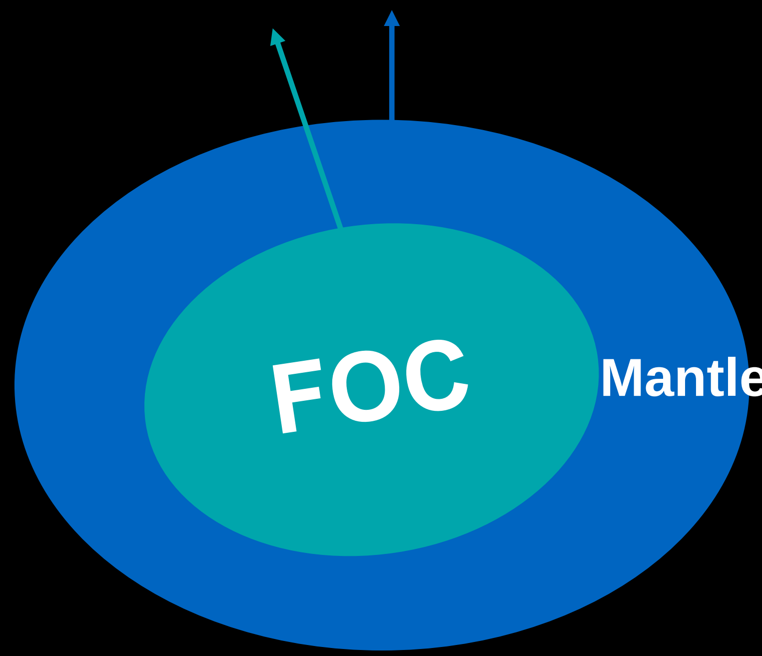
Contents

- **Motivation: FCN**
- **Methodology:**
  - **Multiple layer spectral method: Finite Element Method**
  - **Linear Operator Method**
- **Our FCN results**

Study of Free-core-nutation (FCN)



- FCN is a normal mode of the earth as the rotating axes of the FOC and of the mantle don't coincide。
- a key parameter & key question to be answered:
  - The calculated period of FCN from traditional theory differs largely from the high-precision obs.
  - FCN reflects (depends on) the physics of the FOC, mantle & CM
  - FCN influences strongly the -1yr. nutation due to its resonance.

No fluid OC? → No FCN !

VLBI	430±1
SG(GGP)	430±5
theoretical calculated	458～470

FCN: Assumptions:

- extra flattening at CMB: +5%: too big to be consistent with the overall near-HE shape of the Earth as a whole.
- magnetic/ viscous/topographic couplings @CMB,
- 2nd order (ε^2) terms effect?
- etc.

Contributions of EMC @CMB to nutation (μas)

periods (day)	pro-ip (μas)	pro-op (μas)	retro-ip (μas)	retro-op (μas)
6798.384	+4(+9,+37)	-4(-9,-29)	-20(-83,-328)	+20(+83,+249)
365.260	0(0,-3)	0(0,+3)	-40(-68,-450)	+40(+69,+411)
182.621	+5(+14,+61)	-5(-14,-47)	-2(-3,-16)	+2(+3,+12)
13.661	0(0,+2)	0(0,-1)	0(/,0)	0(/,0)

FCN without EMC = -432.34 solar day (-455.57 sidereal day)

FCN with EMC = -431.96 solar day (-454.39 sidereal day)

change in the FCN period = 0.38 solar day (1.18 sidereal day)

(Huang et al., 2011)

(Buffett,1992)

(Buffett,2002)

MBH2000

➔ Mathews & Guo (2005); Buffett(2011): + viscous

Do /can we really need these various unproved assumptions to get a good FCN ?

FCN,FCN,FCN...

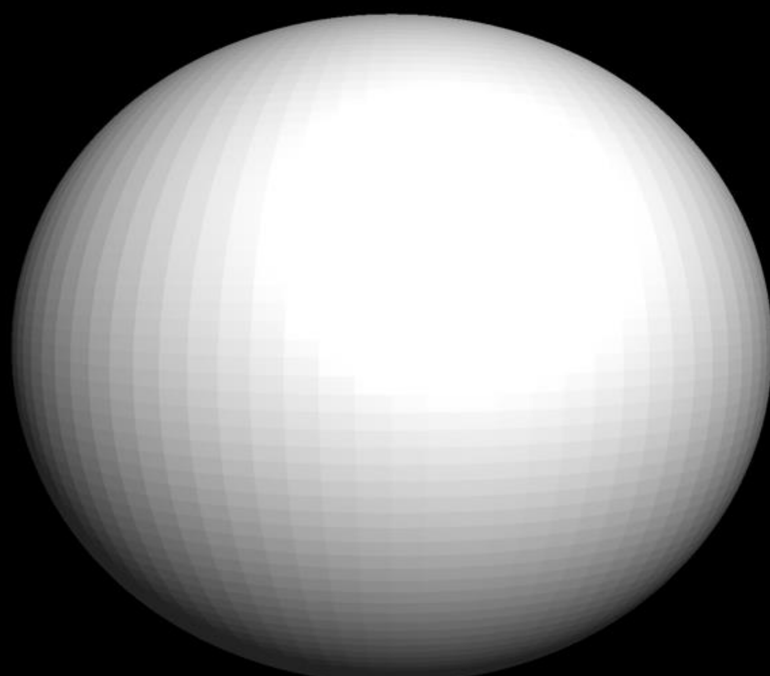
Multiple layer spectral method (MLSM)

+

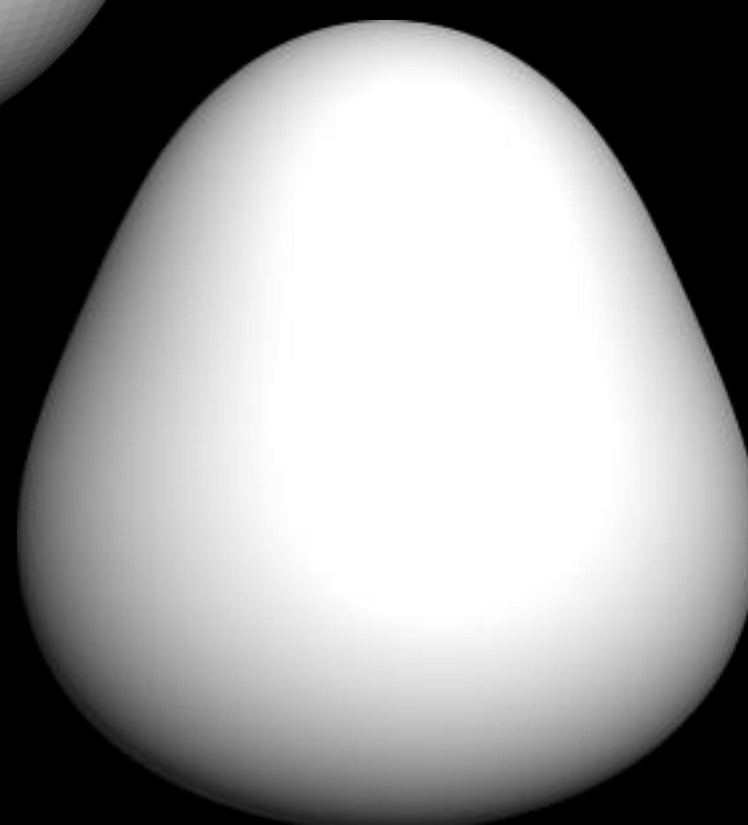
Linear Operator Method (LOM)

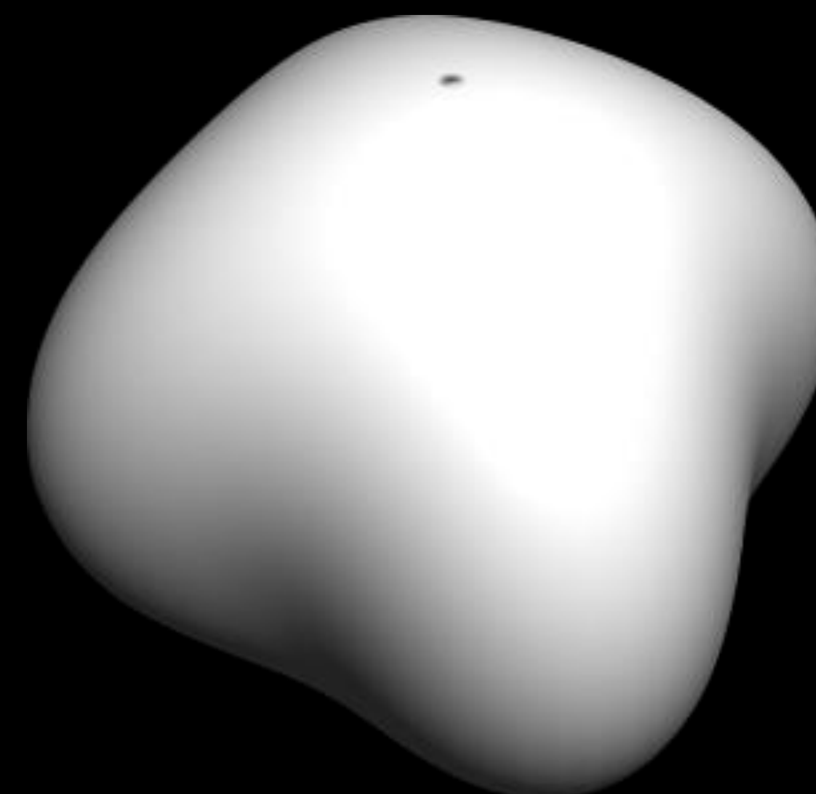
Finite Element Method (FEM)

- Traditional approach solves one order ellipsoid only.
- FEM can solve more complex models.



$$r = 8 - cos^2\theta$$





$$r = 7 - \cos(4\phi) * \sin^4\theta$$

$$r = 6 - 0.5(5 * \cos^3\theta - 3\cos\theta)$$

Main Idea of FEM

Boundary Surface could be described as:

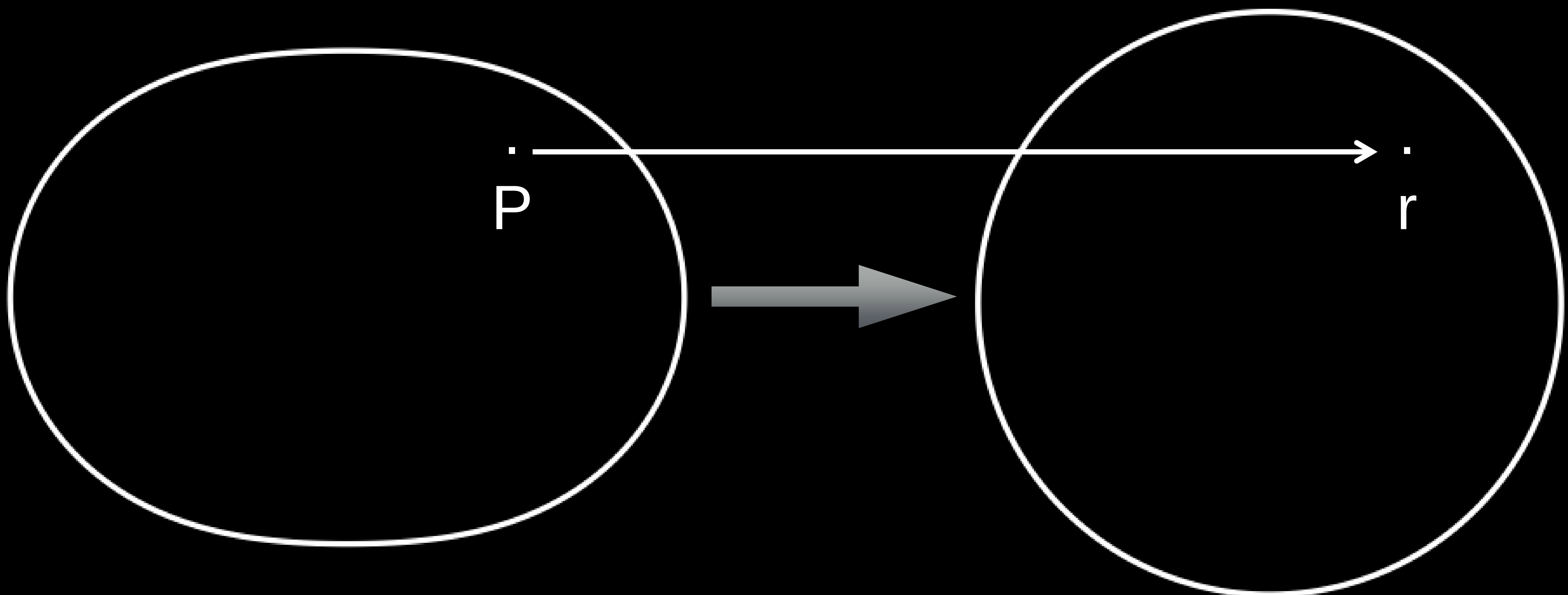
$$r = r_0 + \sum_{n,m} \kappa_n^m Y_n^m(\theta, \phi)$$

Let's consider how to solve the dynamic equation:

$$\rho D_t^2 \vec{u} + 2\rho \vec{\Omega}_0 \times D_t \vec{u} = -\rho \vec{\Omega}_0 \times (\vec{\Omega}_0 \times \vec{u}) + \nabla \cdot \overleftrightarrow{S}^e - \nabla(\gamma \nabla \cdot \vec{u})$$

$$- \rho \nabla \phi_1 - \rho \vec{u} \cdot \nabla\nabla\phi + \nabla \cdot [\gamma (\nabla \vec{u})^T]$$

Traditional Approach: Equivalent Spherical Domain



$$p(r) = r + \hat{r}h(r,\theta)$$

$$h(r,\theta) = -\frac{2}{3}\epsilon(r)P_2(\cos\theta)$$

Traditional Approach: Equations of Simple Example

Governing Equations:

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = \nabla\cdot\overleftrightarrow{S} - \rho(\nabla\cdot\vec{u})\vec{g_0} + \rho\nabla\vec{u}\cdot\vec{g_0} + \rho\vec{u}\cdot\nabla\cdot\vec{g_0} + \rho\nabla V_1$$

$$\overleftrightarrow{S} = \lambda(\nabla\cdot\vec{u})\overleftrightarrow{I} + \mu[\nabla\vec{u} + (\nabla\vec{u})^T]$$

$$\nabla^2 V_1 = 4\pi G\nabla\cdot(\rho\vec{u})$$

Variables:

$$\vec{u}(\vec{r},t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\overrightarrow{S_{nm}}(\vec{r}) + \overrightarrow{T_{nm}}(\vec{r}))e^{i\omega t}$$

$$V_1(\vec{r},t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_{1nm}(r)y_{nm}(\theta,\phi)e^{i\omega t}$$

and

$$\overrightarrow{S_{nm}} = [U_{nm}(r)P_{nm}(\cos\theta)\hat{r} + V_{nm}(r)\frac{d}{d\theta}P_{nm}(\cos\theta)\hat{\theta} + V_{nm}(r)\frac{im}{\sin\theta}P_{nm}(\cos\theta)\hat{\phi}]e^{im\phi}$$

$$\overrightarrow{T_{nm}} = [W_{nm}(r)\frac{im}{\sin\theta}P_{nm}(\cos\theta)\hat{\theta} - W_{nm}(r)\frac{d}{d\theta}P_{nm}(\cos\theta)\hat{\phi}]e^{im\phi}$$

Traditional Approach: Direct numerical integration approach

Variables

$$y_1^n = U_{nm}$$

$$y_2^n = \lambda\Delta_{nm} + 2\mu\frac{dU_{nm}}{dr}$$

$$y_3^n = V_{nm}$$

$$y_4^n = \mu(\frac{dV_{nm}}{dr} + \frac{U_{nm} - V_{nm}}{r})$$

$$y_5^n = V_{1nm}$$

$$y_6^n = \frac{dV_{1nm}}{dr} - 4\pi G\rho U_{nm}$$

$$y_7^n = iW_{nm}$$

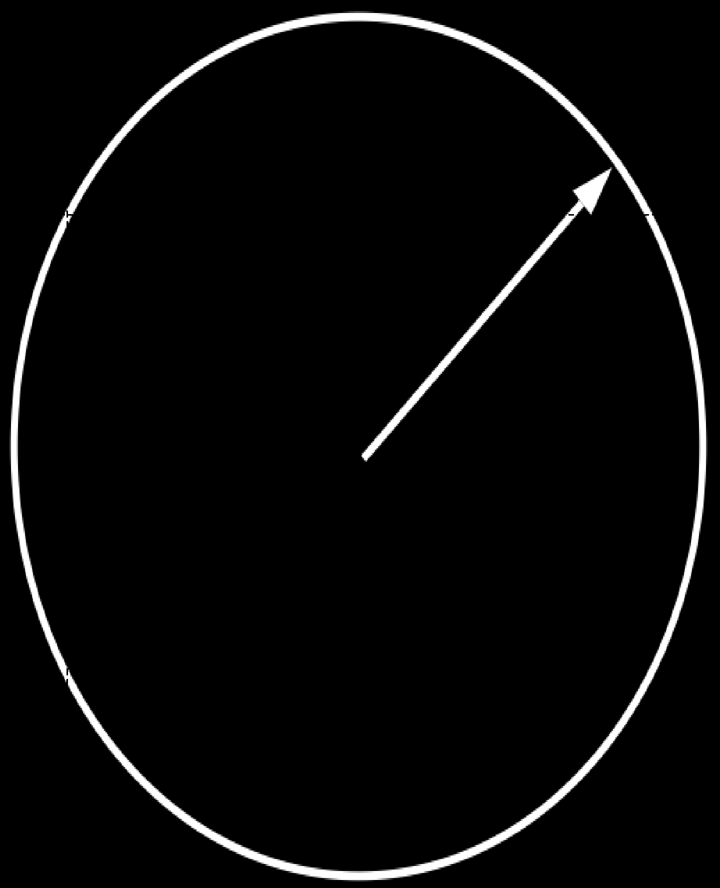
$$y_8^n = i\mu(\frac{dW_{nm}}{dr} - \frac{W_{nm}}{r})$$

Ordinary Differential Equations

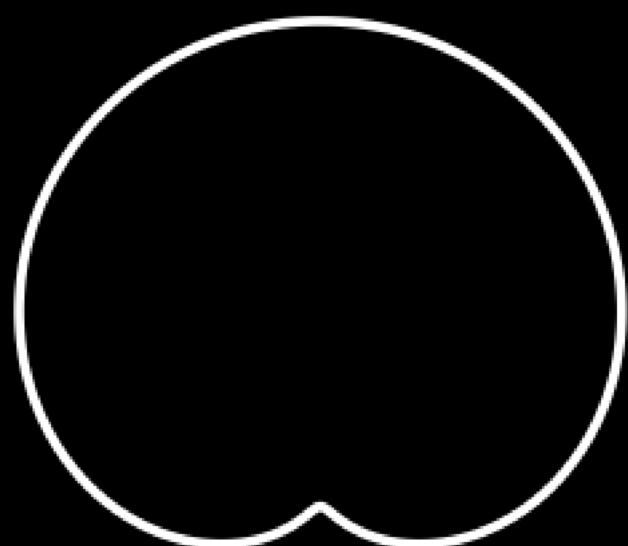
$$\begin{cases}
\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda+2\mu)r}y_1^n + \frac{1}{\lambda+2\mu}y_2^n + \frac{n(n+1)\lambda}{(\lambda+2\mu)r}y_3^n \\
\frac{dy_2^n}{dr} = [-\rho\omega^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2}]y_1^n - \frac{4\mu}{(\lambda+2\mu)r}y_2^n + \\
\qquad [\frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2}]y_3^n + \frac{n(n+1)}{r}y_4^n - \rho y_6^n \\
\frac{dy_3^n}{dr} = -\frac{1}{r}y_1^n + \frac{1}{r}y_3^n + \frac{1}{\mu}y_4^n \\
\frac{dy_4^n}{dr} = [\frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2}]y_1^n - \frac{\lambda}{(\lambda+2\mu)r}y_2^n + \\
\qquad \{-\rho\omega^2 + \frac{2\mu}{(\lambda+2\mu)r^2}[(2n^2+2n-1)\lambda + 2(n^2+n-1)\mu]\}y_3^n \\
\qquad -\frac{3}{r}y_4^n - \frac{\rho}{r}y_5^n \\
\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n \\
\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n
\end{cases}$$

Traditional Approach: Direct numerical integration approach

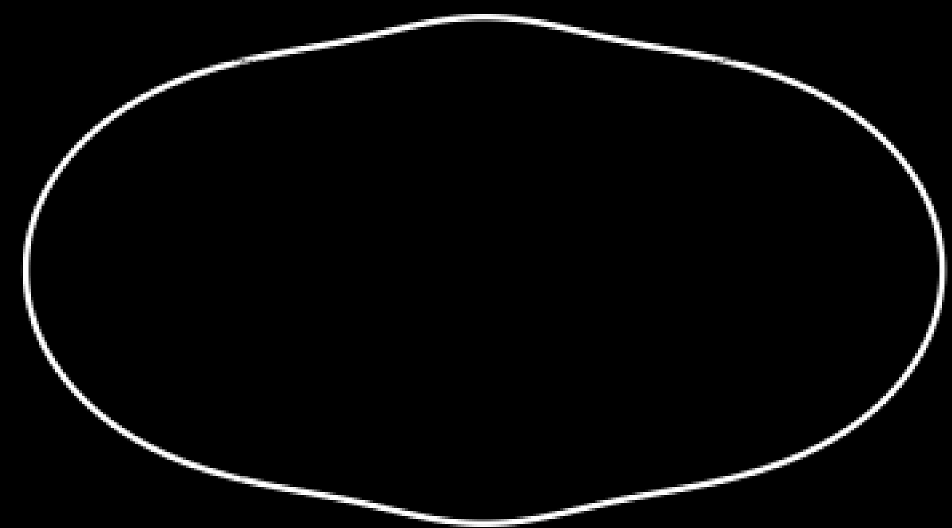
- **Give a possible period and ODEs' initial values**
- **Integrate ODEs from center to surface**
- **Check the integrated values at surface so as to determine whether the possible period is veritable**



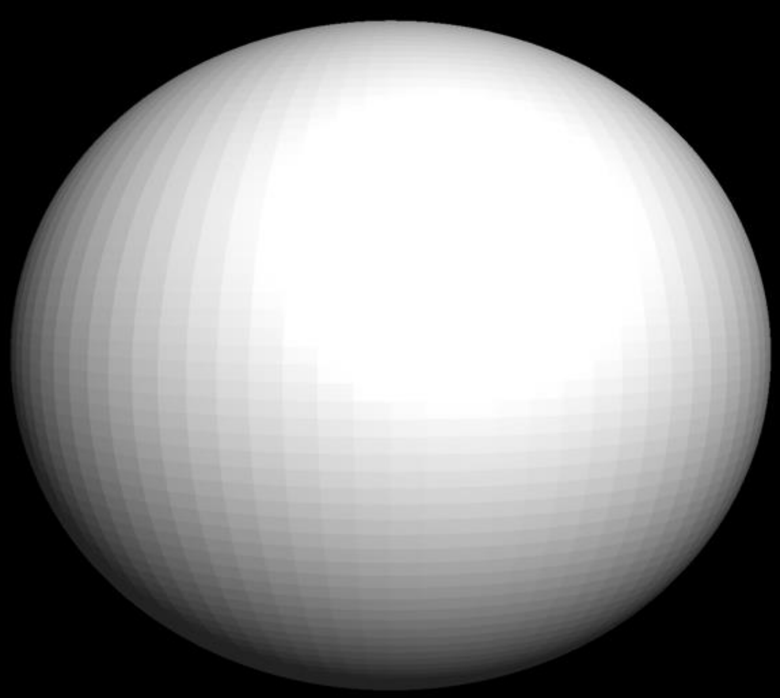
Traditional Approach Problem: the More Complex Figures ? ×



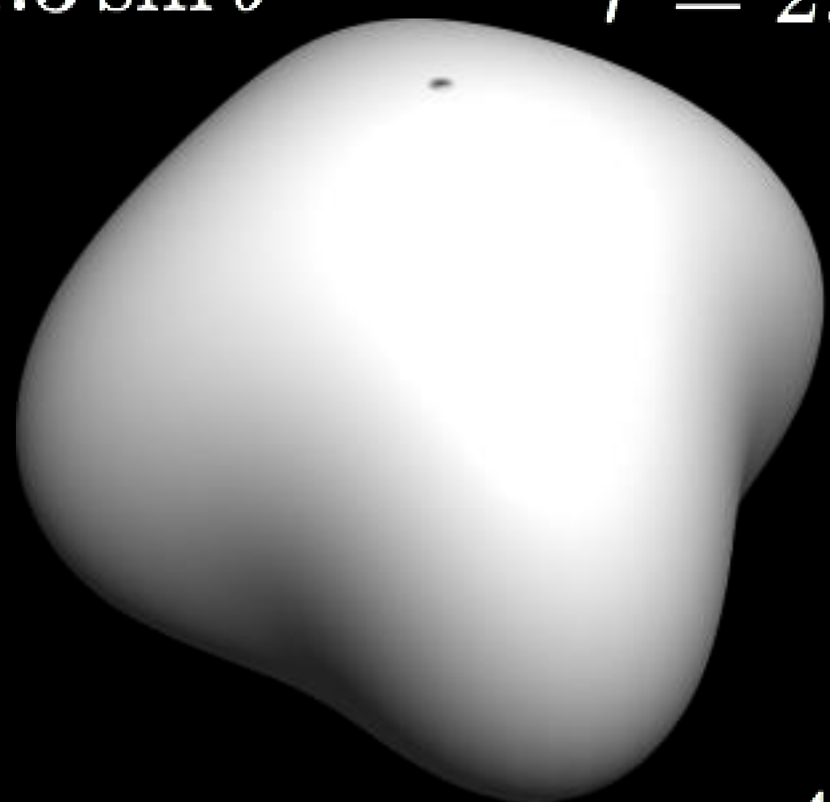
$r = 1.5 + 1.3 \sin\theta$



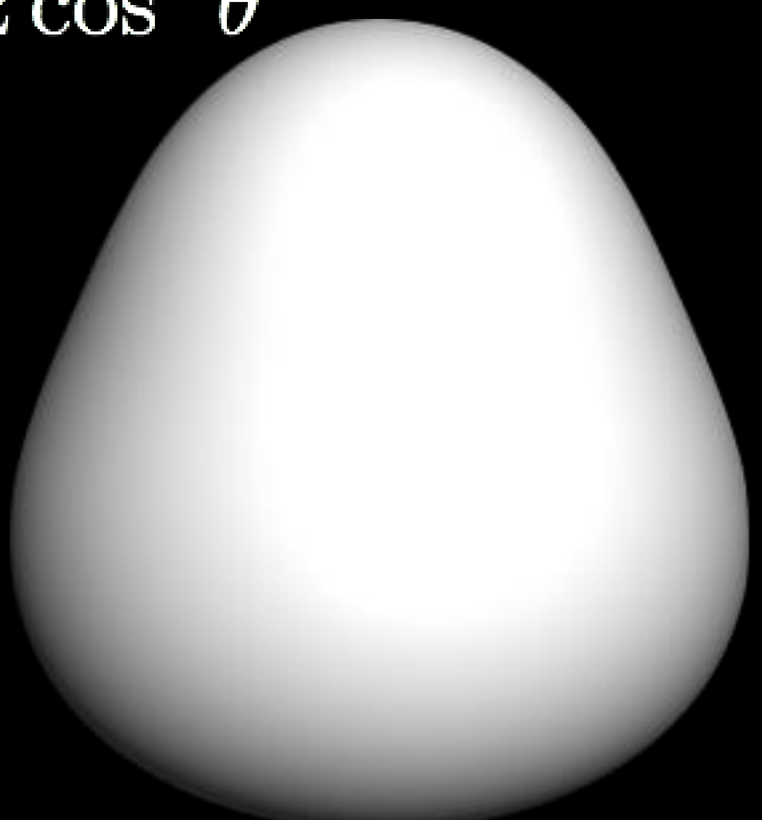
$r = 2.5 + 2\cos^4\theta$



$r = 8 - cos^2\theta$



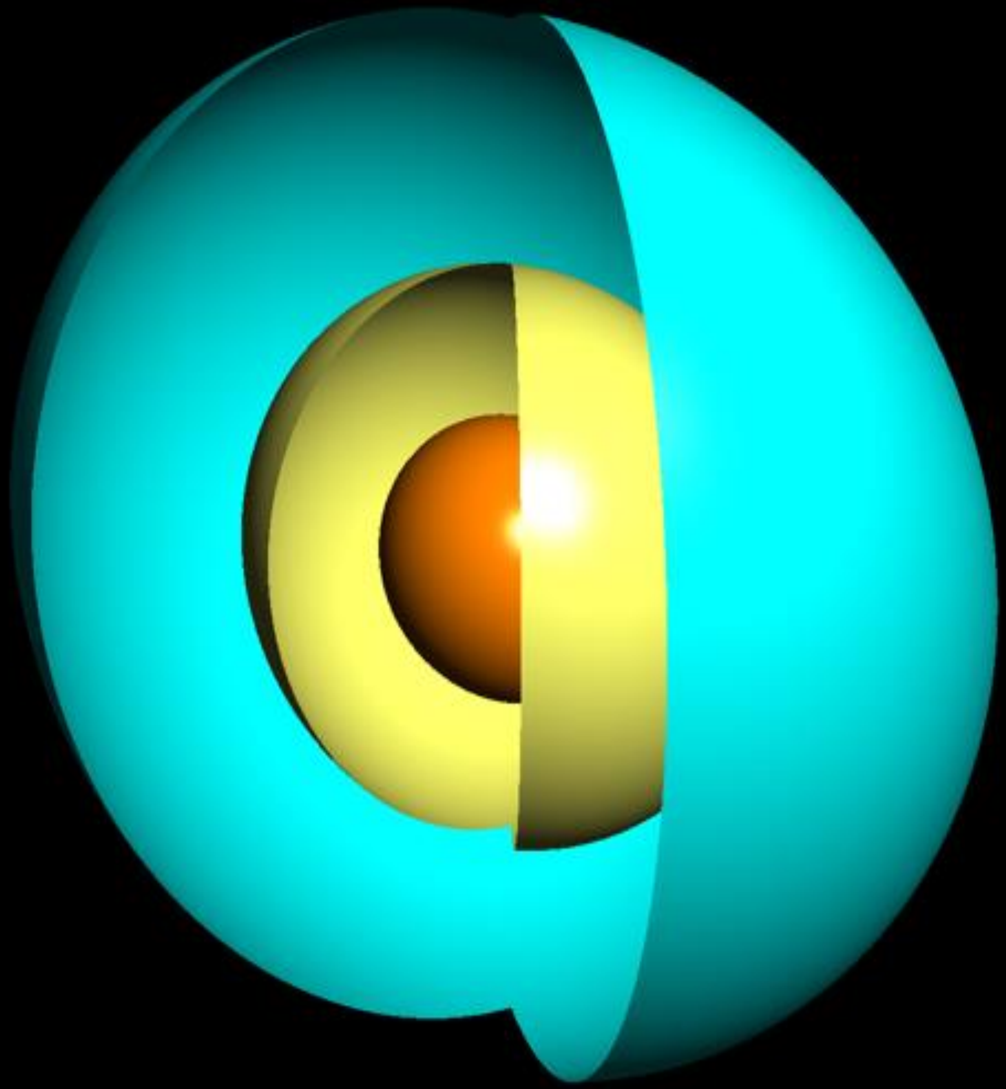
$r = 7 - \cos(4\phi) * \sin^4\theta$



$r = 6 - 0.5(5 * \cos^3\theta - 3\cos\theta)$

FEM: The Whole Domain is separated into several subdomain

- eg, Earth could be separated into 3 subdomains:
  - Solid inner core
  - Fluid outer core
  - Solid mantle (crust)



FEM: Express the Equation in Each Subdomain

The inner and the outer Boundary Surfaces could be described as:

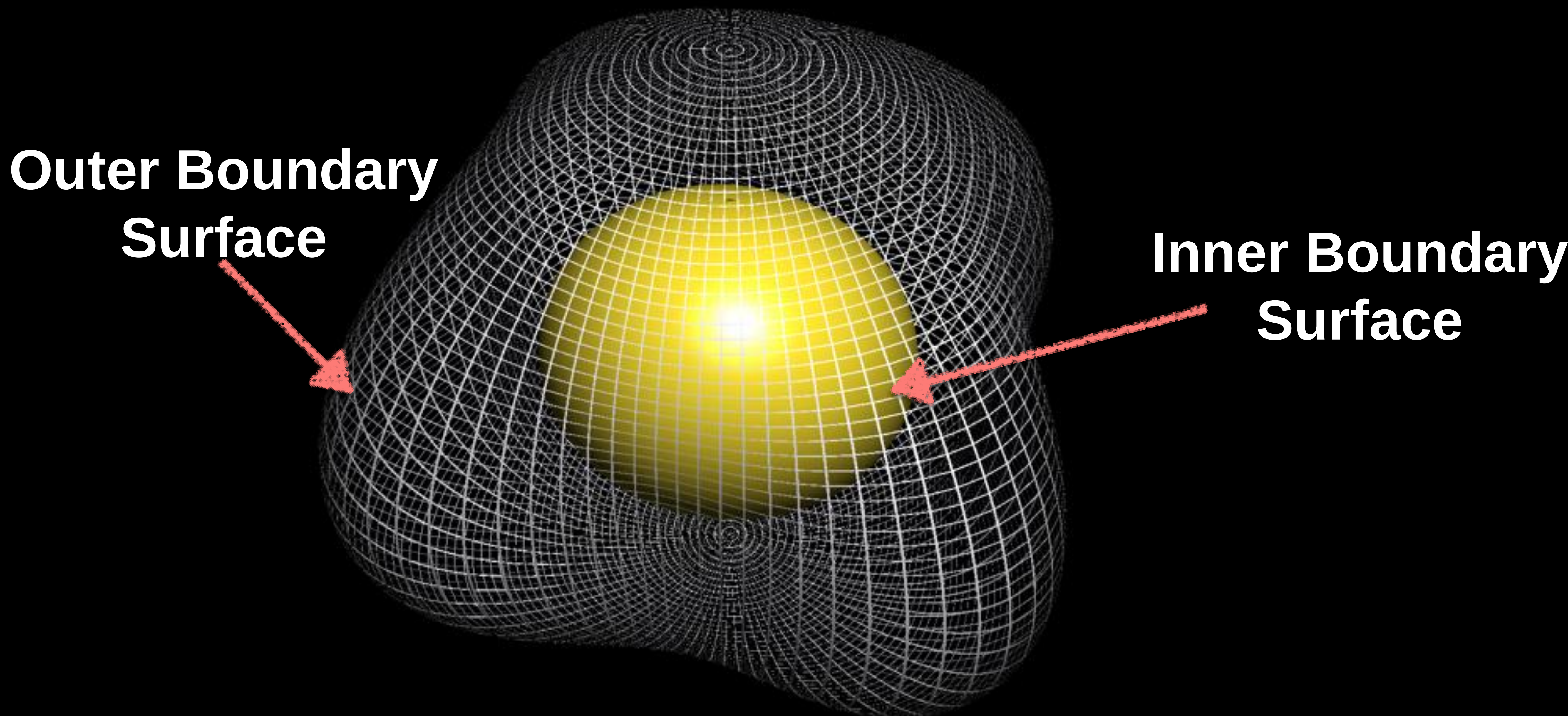
$$r = R_{in} + \sum_{n,m} \xi_n^m Y_n^m(\theta, \phi)$$

$$r = R_{out} + \sum_{n,m} \Xi_n^m Y_n^m(\theta, \phi)$$

Let's consider how to solve the dynamic equation:

$$\rho D_t^2 \overrightarrow{u} + 2\rho \overrightarrow{\Omega}_0 \times D_t \overrightarrow{u} = -\rho \overrightarrow{\Omega}_0 \times (\overrightarrow{\Omega}_0 \times \overrightarrow{u}) + \nabla \cdot \overleftrightarrow{S}^e - \nabla(\gamma \nabla \cdot \overrightarrow{u})$$
$$- \rho \nabla \phi_1 - \rho \overrightarrow{u} \cdot \nabla\nabla\phi + \nabla \cdot [\gamma (\nabla \overrightarrow{u})^T]$$

FEM➔ multiple layer spectral method (MLSM)



multiple layer spectral method (MLSM): Integrate governing eqs. in each subdomain

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{R_{in}+\sum_n^m \xi_n^m Y_n^m(\theta,\phi)}^{R_{out}+\sum_n^m \Xi_n^m Y_n^m(\theta,\phi)} \vec{\Lambda}_{(i,j,k)} \cdot \vec{eq}_v r^2 \sin\theta dr d\theta d\phi = 0$$

Trial vectors(functions)

Dynamic equation

We use MLSM instead of direct numerical integration approach. Variables are expanded in basis function series

MLSM: Galerkin Method

eg, the (2,0) spheroidal displacement field is represented as:

$$\overrightarrow{S}\,^2_0 = \sum_{i=0}^{i_{max}} c_i \cdot \psi_i(r) \cdot \nabla Y_2^0(\theta, \phi)$$

coefficients

Basis function

eg., trial vector(function) of (2,0) spheroidal is:

$$\phi_k(r) \cdot \nabla Y_2^0(\theta, \phi)$$

MLSM: Boundary Conditions

Boundary surface could be described as:

$$r = r_0 + \sum_{n,m} \kappa_n^m Y_n^m(\theta, \phi)$$

The Normal vector of boundary surface is:

$$\vec{n} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}$$

MLSM: Boundary Conditions turn into surface integral

for $\delta\vec{u} = \vec{u}^{+} - \vec{u}^{-} = 0$

$$\oiint \sigma_{(i,j,k)}\hat{n} \cdot \delta\vec{u}dS$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sigma_{(i,j,k)}\{[\hat{r} - \nabla r(\theta,\phi)] \cdot \delta\vec{u}\}[r(\theta,\phi)]^2 \sin\theta d\theta d\phi = 0$$

For scalar continuation condition $\delta a = a^{+} - a^{-} = 0$

$$\oiint |\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}| \sin\theta * \delta a * |\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}| dS$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} |\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}|^2 * \delta a * \sin\theta * d\theta d\phi = 0$$

MLSM: Combine All Equations

- Combine these equations so as to build a matrix:
  - Volume integral of governing equations in each subdomain
  - Surface integral of boundary conditions between 2 adjoint subdomain
  - Free surface boundary condition
- No need for the initial value at center. As center condition just require parameters to be reasonable which is the absence of r or r^2 basis terms.

MLSM: Search the Period

- Pick an period and compute the condition number of the matrix.

Linear Operator Method (LOM)

- Why Use Linear Operator method?
  - Generalized Spherical Harmonics(GSH) are a little bit abstruse. It needs knowledge of group theory and representation theory.
  - Boundary conditions could be easily solved.

Linear Operator Method (LOM)

- **Equations is based on spherical harmonics (SH) with unified form:**

$$Y_n^m(\theta,\phi) = \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n,m,s)(\cos\theta)^{n-2s-|m|}(\sin\theta e^{\zeta(m)i\phi})^{|m|}$$

- **Each SH can be built up by 3 atoms:**

$$\cos\theta, \sin\theta e^{i\phi}, \sin\theta e^{-i\phi}$$

- **If we know 3 atoms' actions on each other, all computation about SH are obtained.**

LOM Example: Product of two Spherical Harmonics

- As we know:

$$\cos\theta * Y_n^m(\theta,\phi) = d_1(n,m) * Y_{n+1}^m(\theta,\phi) + d_2(n,m) * Y_{n-1}^m(\theta,\phi)$$

$$\sin\theta e^{i\phi} * Y_n^m(\theta,\phi) = d_{10}(n,m) Y_{n+1}^{m+1}(\theta,\phi) + d_{11}(n,m) Y_{n-1}^{m+1}(\theta,\phi)$$

$$\sin\theta e^{-i\phi} * Y_n^m(\theta,\phi) = d_{12}(n,m) Y_{n+1}^{m-1}(\theta,\phi) + d_{13}(n,m) Y_{n-1}^{m-1}(\theta,\phi)$$

- By

$$Y_n^m(\theta,\phi) = \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n,m,s)(\cos\theta)^{n-2s-|m|}(\sin\theta e^{\zeta(m)i\phi})^{|m|}$$

- We can get

$$Y_n^m(\theta,\phi) * Y_i^j(\theta,\phi)$$

LOM Example: Product of a SH and a VSH

- As we know:

$$\cos\theta * \overrightarrow{S}_n^m(\theta,\phi) = d_7(n,m) * \overrightarrow{S}_{n+1}^m(\theta,\phi) + d_8(n,m) * \overrightarrow{S}_{n-1}^m(\theta,\phi) + d_9(n,m) * \overrightarrow{T}_n^m(\theta,\phi)$$

$$\sin\theta e^{i\phi} * \overrightarrow{S}_n^m(\theta,\phi) = d_{14}(n,m) * \overrightarrow{S}_{n+1}^{m+1}(\theta,\phi) + d_{15}(n,m) * \overrightarrow{S}_{n-1}^{m+1}(\theta,\phi) + d_{16}(n,m) * \overrightarrow{T}_n^{m+1}(\theta,\phi)$$

$$\sin\theta e^{-i\phi} * \overrightarrow{S}_n^m(\theta,\phi) = d_{17}(n,m) * \overrightarrow{S}_{n+1}^{m-1}(\theta,\phi) + d_{18}(n,m) * \overrightarrow{S}_{n-1}^{m-1}(\theta,\phi) + d_{19}(n,m) * \overrightarrow{T}_n^{m-1}(\theta,\phi)$$

- We can get

$$Y_a^b(\theta,\phi) * \nabla_1 Y_n^m(\theta,\phi)$$

LOM Example: Dot-product of two VSH

Vector Spheroidal harmonics （VSH) can be written in following form:

$$\nabla_1 Y_n^m(\theta,\phi) = H_0[Y_n^m(\theta,\phi)] * \nabla_1 \cos\theta + H_1[Y_n^m(\theta,\phi)] * \nabla_1(\sin\theta e^{\zeta(m)i\phi})$$

- H_0 and H_1 are combination of SH:

$$H_0[Y_n^m(\theta,\phi)] = \sum_{s=0}^{[(n-|m|)/2]} (n-2s-|m|) * c_{10}(n,m,s)(\sin\theta e^{\zeta(m)i\phi})^{|m|} * (\cos\theta)^{n-2s-|m|-1}$$

$$H_1[Y_n^m(\theta,\phi)] = \sum_{s=0}^{[(n-|m|)/2]} |m| * c_{10}(n,m,s)(\cos\theta)^{n-2s-|m|} * (\sin\theta e^{\zeta(m)i\phi})^{|m|-1}$$

LOM Example: Dot-product of two VSH

- If we get:

$$\nabla_1 \cos\theta \cdot \nabla_1 Y_a^b(\theta,\phi)$$

$$\nabla_1(\sin\theta e^{i\phi}) \cdot \nabla_1 Y_a^b(\theta,\phi)$$

$$\nabla_1(\sin\theta e^{-i\phi}) \cdot \nabla_1 Y_a^b(\theta,\phi)$$

- We can finally get:

$$\nabla_1 Y_n^m(\theta,\phi) \cdot \nabla_1 Y_a^b(\theta,\phi)$$

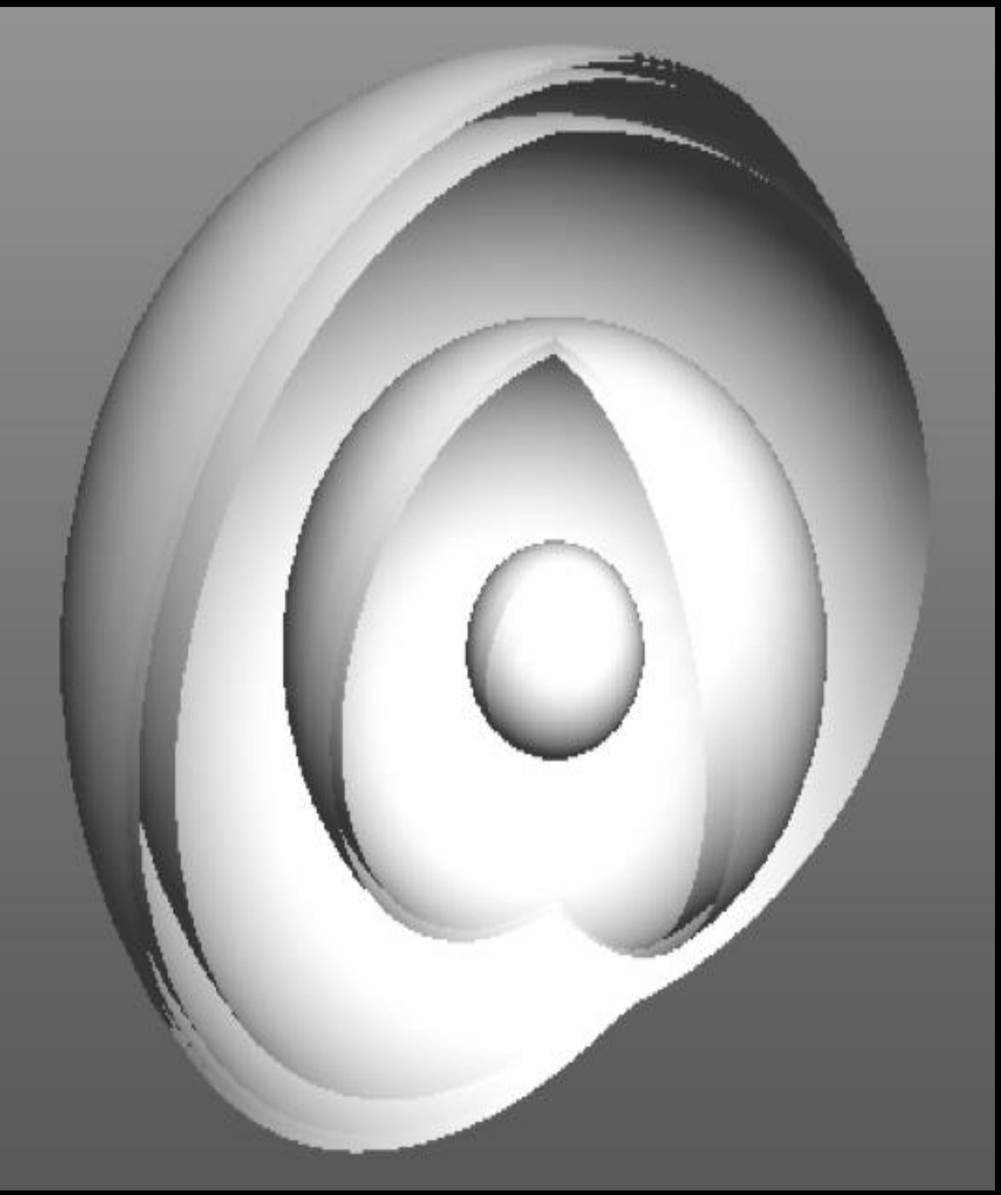
LOM

- **Use this method, we get**
  - **product of**
    - **2 SH**
    - **a SH & a VSH**
  - **dot product of 2 VSH**
  - **cross product of 2 VSH**
  - **gradient of a SH**
  - **curl of a VSH**
  - **divergence of a VSH**

LOM: Tensors

- **It is difficult to represent the stress tensor in a stand-alone form. But in the equation it only needs the divergence of the tensor, while in boundary conditions it only needs the dot-product of the normal vector and the tensor, and these two terms can be represented by the LOM.**

Our Preliminary Earth Model



- **PREM**
- **1 order ellipticity**
- **3 layers**
  - **without ocean**
  - **1 layer for solid inner core**
  - **1 layer for fluid outer core**
  - **1 layer for mantle and crust (while 10 layers of parameters)**

Validation: Tilt-Over-Mode

- **The displacement field is truncated as:**

$$\vec{T}_1^1 + \vec{S}_2^1 + \vec{T}_3^1$$

- **Each term is expanded in polynomial series, in each subdomain**

$$\vec{T}_1^1 = \sum_{i=0}^{Max-order} c_i * r^i * [\hat{r} \times \nabla_1 Y_1^1(\theta,\phi)]$$

Order of Polynomial in trial function

i_{max}	TOM
2	1+1.8e-2
3	1+1.8e-3
5	1+1.5e-4
8	1+2.0e-5

FCN Result

- The displacement field is truncated as:

$$\vec{T}_1^1 + \vec{S}_2^1 + \vec{T}_3^1$$

- FCN is very sensitive to the ellipticity at CMB. Our value is equivalent to the most authors': $2.54656*10^{-3}$.

- Result converges when polynomial order i_{max} >= 4.

VLBI	430±1
SG(GGP)	430±5
Theory calculated	458～470
Our Approach	435±3

Discussion：Why MLSM is Better?

- **avoiding derivatives of some parameters which are not precise in earth model.**

$$\partial_r(\rho, \lambda, \mu)$$

$$\lambda(P) = \lambda_0(P) + \frac{2}{3}\epsilon(P)p\frac{\partial\lambda_0(P)}{\partial p}P_2(\cos\theta)$$

$$\boxed{\int_{\mathrm{r}} \partial_r(\rho, \lambda, \mu)dr \Rightarrow (\rho, \lambda, \mu)}$$

- **MLSM focuses on global characteristics**

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{R_{in}+\sum_n^m \xi_n^m Y_n^m(\theta,\phi)}^{R_{out}+\sum_n^m \Xi_n^m Y_n^m(\theta,\phi)} \overrightarrow{\Lambda}_{(i,j,k)} \cdot \overrightarrow{eq}_v r^2 \sin\theta dr d\theta d\phi = 0$$

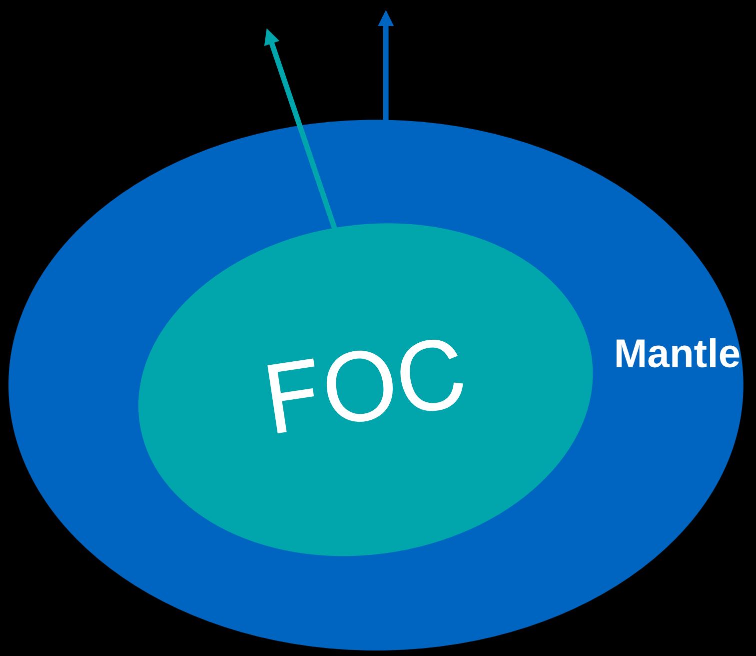
Discussion & Next Step?

- **Clairaut coordinates (Rochester et al. ,2014): =458**
- **2nd order (ε^2) terms effect ?**
- **truncated coupling chain:** $\overrightarrow{T}_1^1 + \overrightarrow{S}_2^1 + \overrightarrow{T}_3^1$ **+… ?**
- **Rotational modes of Jupiter, Saturn & exoplanet**

Thanks!

Study of Free-core-nutation (FCN)

- FCN is a normal mode of the earth as the rotating axes of the FOC and of the mantle don't coincident。
- FCN depends on the physics of the FOC, mantle & core-mantle-boundary. It influences strongly the retro-annual nutation due to its resonance, so it is a key parameter & interesting topic.
- The calculated period of FCN from traditional theory differs largely from the high-precision observation.
- We developed an integrated Galerkin method and spectral element method that can study any antisymmetric earth without GSH.
- These methods are applied on the computation of FCN period from PREM earth without any assumption (eg., extra flattening at CMB, magnetic/ viscous/topographic couplings at CMB, etc.).
- Our result is 435 sid. day !



No fluid OC? ➔ No FCN !

Obs.(VLBI+SG)	430±1
Theoretically Calculated	458-465
This work	435

(Zhang & Huang, 2014a,b, c)