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EXPANSION OF THE HAMILTONIAN OF A PLANETARY SYSTEM

INTO THE POISSON SERIES IN ALL ELEMENTS

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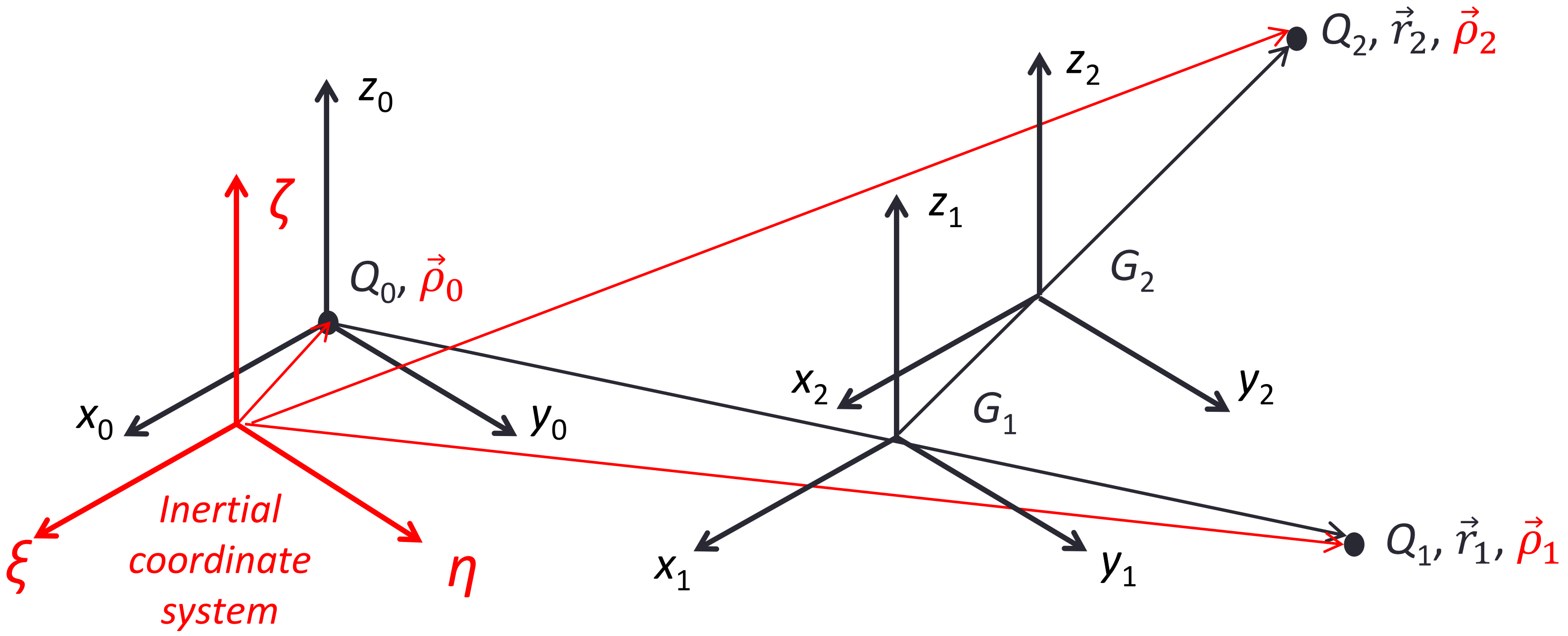
INTRODUCTION

Jacobi coordinate system



body Q_0 has mass m_0,
body Q_i has mass $\mu m_0 m_i$ $(i = 1 \ldots 4)$,
μ – small parameter

G_i – barycenters of subsystems
$\vec{r}_i$ – Jacobi radius vectors
$\vec{\rho}_i$ – Inertial radius vectors

Converting between Inertial and Jacobi coordinates

From Inertial to Jacobi

$$\vec{r}_0 = \frac{1}{\overline{m}_N}\vec{\rho}_0 + \mu\sum_{k=1}^{N}\frac{m_k}{\overline{m}_N}\vec{\rho}_k$$

$$\vec{r}_i = \vec{\rho}_i - \frac{1}{\overline{m}_{i-1}}\vec{\rho}_0 - \frac{\mu}{\overline{m}_{i-1}}\sum_{k=1}^{i-1} m_k\,\vec{\rho}_k$$

From Jacobi to Inertial

$$\vec{\rho}_0 = \vec{r}_0 - \mu\sum_{k=1}^{N}\frac{m_k}{\overline{m}_N}\vec{r}_k$$

$$\vec{\rho}_i = \vec{r}_0 + \frac{\overline{m}_{i-1}}{\overline{m}_i}\vec{r}_i - \frac{\mu}{\overline{m}_{i-1}}\sum_{k=1}^{N}\frac{m_k}{\overline{m}_k}\vec{r}_k$$

$\overline{m}_i = 1 + \mu\sum m_k$ – sum of masses

μ – small parameter

for Solar system
μ = 0.001

Differences between coordinates

$$\vec{\rho}_i - \vec{\rho}_0 = \vec{r}_i + \mu\sum_{k=1}^{i-1}\frac{m_k}{\overline{m}_k}\vec{r}_k$$

$$\vec{\rho}_i - \vec{\rho}_j = \vec{r}_i - \vec{r}_j + \mu\sum_{k=j}^{i-1}\frac{m_k}{\overline{m}_k}\vec{r}_k$$

The Hamiltonian

$$\boxed{h = h_0 + \mu h_1}$$

$$h_0 = -\frac{Gm_0}{2}\sum_{k=1}^{N}\frac{m_k}{a_k} \qquad h_1 = Gm_0\left[\sum_{i=2}^{N}\frac{m_i}{\mu}\left(\frac{1}{r_i} - \frac{1}{|\vec{\rho}_i - \vec{\rho}_0|}\right) - \sum_{i=1}^{N}\sum_{j=1}^{i-1}\frac{m_i m_j}{|\vec{\rho}_i - \vec{\rho}_j|}\right]$$

Introduce next notations: $\vec{R}_i = \sum_{k=1}^{i-1}\frac{m_k}{\overline{m}_k}\vec{r}_k\,,\ \tilde{R}_i = \sqrt{\vec{r}_i^{\,2} + 2\mu\vec{r}_i\vec{R}_i + \mu^2\vec{R}_i^{\,2}}.$

Then: $\vec{\rho}_i - \vec{\rho}_0 = \vec{r}_i + \mu\vec{R}_i,\quad |\vec{\rho}_i - \vec{\rho}_0| = \tilde{R}_i$ and if $h_1 = \frac{Gm_0}{a_0}h_2$ get

$$\boxed{h_2 = \sum_{i=2}^{N}\frac{a_0 m_i\left(2\vec{r}_i\vec{R}_i + \mu\vec{R}_i^{\,2}\right)}{r_i\tilde{R}_i\left(r_i + \tilde{R}_i\right)} - \sum_{i=1}^{N}\sum_{j=1}^{i-1}\frac{m_i m_j}{|\vec{\rho}_i - \vec{\rho}_j|}}$$

Disturbing function

Second system of Poincare elements

Poincare orbital elements:

- $L = \varkappa M \sqrt{a}$
- $\lambda = l + \omega + \Omega$
- $\xi_1 = \sqrt{2\varkappa M \sqrt{a}\left(1 - \sqrt{1 - e^2}\right)} \cos(\omega + \Omega\ )$
- $\eta_1 = -\sqrt{2\varkappa M \sqrt{a}\left(1 - \sqrt{1 - e^2}\right)} \sin(\omega + \Omega\ )$

eccentric elements

$\xi_1, \eta_1 \sim e$

- $\xi_2 = \sqrt{2\varkappa M \sqrt{a}\sqrt{1 - e^2}(1 - \cos i)} \cos \Omega$
- $\eta_2 = -\sqrt{2\varkappa M \sqrt{a}\sqrt{1 - e^2}(1 - \cos i)} \sin \Omega$

oblique elements

$\xi_2, \eta_2 \sim i$

where *a, e, i, ω, Ω, l* are Kepler orbital elements,

$\varkappa^2$ is gravitational parameter, M is normalized mass

Expansion of the Hamiltonian

- The Hamiltonian can be written in the Poisson series in the following form:

$$h = h_0 + \sum A_{kn} x^k \mathrm{e}^{\mathrm{i} n\lambda}$$

  - h_0 – undisturbed Hamiltonian
  - A_{kn} – numerical coefficients
  - $x^k = L_1{}^{k_1} \xi_{1,1}{}^{k_2} \eta_{1,1}{}^{k_3} \xi_{2,1}{}^{k_4} \eta_{2,1}{}^{k_5} \ldots L_4{}^{k_{16}} \xi_{1,4}{}^{k_{17}} \eta_{1,4}{}^{k_{18}} \xi_{2,4}{}^{k_{19}} \eta_{2,4}{}^{k_{20}}$
  - $n\lambda = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 + n_4\lambda_4$

- Computer algebra system “Piranha”[1] was used for this.

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[1] Biscani F. The Piranha algebraic manipulator. – 2009. – p.24. arXiv: 0907.2076v1.

ALGORITHM

Expansion of general part of disturbing function

$$Expansion\ of\ \ |\vec{\rho}-\vec{\rho}'|^{-1}$$

$$\vec{\rho}_i-\vec{\rho}_j=\vec{r}_i-\vec{r}_j+\mu\sum_{k=j}^{i-1}\frac{m_k}{\overline{m}_k}\vec{r}_k$$

$$\frac{1}{|\vec{\rho}_i-\vec{\rho}_j|}=\frac{1}{|\vec{r}_i-\vec{r}_j|}\left(1+\frac{2\mu(\vec{r}_i-\vec{r}_j)\sum\left(\frac{m_k}{\overline{m}_k}\vec{r}_k\right)+\mu^2\left(\sum\left(\frac{m_k}{\overline{m}_k}\vec{r}_k\right)\right)^2}{|\vec{r}_i-\vec{r}_j|^2}\right)^{-\frac{1}{2}}=$$

$$=\frac{1}{\Delta}-\mu\frac{S_{ij}}{\Delta^3}+\mu^2\left(-\frac{1}{2}\frac{R_{ij}}{\Delta^3}+\frac{3}{2}\frac{{S_{ij}}^2}{\Delta^5}\right)+\cdots$$

$$\Delta=|\vec{r}_i-\vec{r}_j|,\quad S_{ij}=(\vec{r}_i-\vec{r}_j)\sum\left(\frac{m_k}{\overline{m}_k}\vec{r}_k\right),\quad R_{ij}=\left(\sum\left(\frac{m_k}{\overline{m}_k}\vec{r}_k\right)\right)^2$$

Expansion of general part of disturbing function

$$Expansion\ of\ \frac{1}{\Delta} = |\vec{r} - \vec{r}'|^{-1}$$

$$\Delta^2 = (\vec{r} - \vec{r}')^2 = r^2 + r'^2 - 2rr' \cos H$$

$$\boxed{\frac{1}{\Delta} = \frac{1}{r'} \underbrace{(1 + \rho^2 - 2\rho \cos H)^{-\frac{1}{2}}}_{\text{is}} = \frac{1}{r'} \sum_{n=0}^{\infty} \rho^n P_n(\cos H)}$$

a generating function
of Legendre polynomials

- $\rho = \frac{r}{r'}$
- $P_n(\cos H)$ – Legendre polynomial of degree n
- H – angle between vectors $\vec{r}$ and $\vec{r}'$

Expansion of second part of disturbing function

$$\frac{(2S+\mu R)}{r\tilde{R}\left(r+\tilde{R}\right)}=\frac{1}{r^3}\frac{2S'+\mu R}{\sqrt{1+\frac{2\mu S'+\mu^2 R}{r^2}}\left(1+\sqrt{1+\frac{2\mu S'+\mu^2 R}{r^2}}\right)}=$$

$$=\frac{S'}{r^3}+\mu\left(\frac{1}{2}\frac{R}{r^3}+\frac{3}{2}\frac{S'^2}{r^5}\right)+\mu^2\left(\frac{3}{2}\frac{RS'}{r^5}+\frac{5}{2}\frac{S'^3}{r^7}\right)+\ldots$$

$$S'=\sum\left(\vec{r}_i\vec{r}_j\right),\qquad R=\left(\sum\left(\frac{m_k}{\overline{m}_k}\vec{r}_k\right)\right)^2$$

- For expansion of scalar products and radius vectors we used classical expansions of Celestial mechanics for $\frac{x}{a}, \frac{y}{a}, \frac{z}{a}, \frac{r}{a}, \frac{a}{r}$.

Basic expansions of $\frac{x}{a}, \frac{y}{a}, \frac{z}{a}$

- $\frac{x}{a} = -\frac{3}{2}L^{-\frac{1}{2}}\xi_1 + L^{-\frac{3}{2}}\left(\frac{3}{16}\xi_1{}^3 + \frac{3}{16}\eta_1{}^2\xi_1 + \frac{3}{4}\eta_2{}^2\xi_1 - \frac{3}{4}\eta_1\xi_2\eta_2\right) +$
  $+\cos\lambda\left(1 - L^{-1}\left(\frac{1}{2}\eta_2{}^2 + \frac{5}{8}\eta_1{}^2 + \frac{3}{8}\xi_1{}^2\right) + \cdots\right) + \sin\lambda\left(L^{-1}\left(\frac{1}{4}\xi_1\eta_1 - \frac{1}{2}\xi_2\eta_2\right) + \cdots\right) +$
  $+\cos 2\lambda\left(\frac{1}{2}L^{-\frac{1}{2}}\xi_1 - L^{-\frac{3}{2}}\left(\frac{19}{48}\xi_1{}^3 + \frac{1}{4}\xi_1\eta_2{}^2 + \frac{9}{16}\xi_1\eta_1{}^2 + \frac{1}{4}\eta_1\xi_2\eta_2\right) + \cdots\right) +$
  $+\sin 2\lambda\left(-\frac{1}{2}L^{-\frac{1}{2}}\eta_1 + L^{-\frac{3}{2}}\left(\frac{23}{48}\eta_1{}^3 + \frac{1}{4}\eta_1\eta_2{}^2 + \frac{5}{16}\eta_1\xi_1{}^2 - \frac{1}{4}\xi_1\xi_2\eta_2\right) + \cdots\right) + \cdots$
- $\frac{y}{a} = \frac{3}{2}L^{-\frac{1}{2}}\eta_1 - L^{-\frac{3}{2}}\left(\frac{3}{16}\eta_1{}^3 + \frac{3}{16}\xi_1{}^2\eta_1 + \frac{3}{4}\xi_2{}^2\eta_1 - \frac{3}{4}\xi_1\xi_2\eta_2\right) +$
  $+\cos\lambda\left(L^{-1}\left(-\frac{1}{4}\xi_1\eta_1 - \frac{1}{2}\xi_2\eta_2\right) + \cdots\right) + \sin\lambda\left(1 - L^{-1}\left(\frac{1}{2}\xi_2{}^2 + \frac{3}{8}\eta_1{}^2 + \frac{5}{8}\xi_1{}^2\right) + \cdots\right) +$
  $+\cos 2\lambda\left(\frac{1}{2}L^{-\frac{1}{2}}\xi_1 - L^{-\frac{3}{2}}\left(\frac{19}{48}\eta_1{}^3 + \frac{1}{4}\eta_1\xi_2{}^2 + \frac{9}{16}\eta_1\xi_1{}^2 + \frac{1}{4}\xi_1\xi_2\eta_2\right) + \cdots\right) +$
  $+\sin 2\lambda\left(\frac{1}{2}L^{-\frac{1}{2}}\eta_1 - L^{-\frac{3}{2}}\left(\frac{23}{48}\xi_1{}^3 + \frac{1}{4}\xi_1\xi_2{}^2 + \frac{5}{16}\eta_1\xi_1{}^2 - \frac{1}{4}\eta_1\xi_2\eta_2\right) + \cdots\right) + \cdots$
- $\frac{z}{a} = \frac{3}{2}L^{-1}(\xi_2\eta_1 - \xi_1\eta_2) + \cos\lambda\left(L^{-\frac{1}{2}}\eta_2 - \frac{1}{8}L^{-\frac{3}{2}}(\eta_2{}^3 + \xi_2{}^2\eta_2 + \xi_1{}^2\eta_2 + 3\eta_1{}^2\eta_2 + \xi_2\xi_1\eta_1) + \cdots\right) +$
  $+\sin\lambda\left(L^{-\frac{1}{2}}\xi_2 - \frac{1}{8}L^{-\frac{3}{2}}(\xi_2{}^3 + \eta_2{}^2\xi_2 + \eta_1{}^2\xi_2 + 3\xi_1{}^2\xi_2 + \eta_2\eta_1\xi_1) + \cdots\right) +$
  $+\cos 2\lambda\left(\frac{1}{2}L^{-1}(\xi_2\eta_1 + \xi_1\eta_2)\right) + \sin 2\lambda\left(\frac{1}{2}L^{-1}(\xi_1\xi_2 - \eta_1\eta_2)\right) + \cdots$

Basic expansions of $\frac{r}{a}, \frac{a}{r}$

- $$\begin{aligned}\frac{r}{a} = {} & 1 + \tfrac{1}{2}L^{-1}({\xi_1}^2 + {\eta_1}^2) - \tfrac{1}{4}L^{-2}\left({\xi_1}^2{\eta_1}^2 + \tfrac{1}{2}({\xi_1}^4 + {\eta_1}^4)\right) + \\ & + \cos\lambda\left(-L^{-\frac{1}{2}}\xi_1 + \tfrac{1}{2}L^{-\frac{3}{2}}({\eta_1}^2\xi_1 + {\xi_1}^3) + \cdots\right) + \sin\lambda\left(-L^{-\frac{1}{2}}\eta_1 - \tfrac{1}{2}L^{-\frac{3}{2}}({\xi_1}^2\eta_1 + {\eta_1}^3) + \cdots\right) + \\ & + \cos 2\lambda\left(\tfrac{1}{2}L^{-1}({\xi_1}^2 - {\eta_1}^2) + \tfrac{11}{24}L^{-2}({\xi_1}^4 - {\eta_1}^4) + \cdots\right) + \sin 2\lambda\left(L^{-1}\xi_1\eta_1 - \tfrac{11}{12}L^{-2}({\eta_1}^3\xi_1 + \eta_1{\xi_1}^3) + \cdots\right) + \\ & + \cos 3\lambda\left(\tfrac{3}{8}L^{-\frac{3}{2}}(3{\eta_1}^2\xi_1 - {\xi_1}^3) + \cdots\right) + \sin 3\lambda\left(\tfrac{3}{8}L^{-\frac{3}{2}}(3{\xi_1}^2\eta_1 - {\eta_1}^3) + \cdots\right) + \\ & + \cos 4\lambda\left(L^{-2}\left(2{\xi_1}^2{\eta_1}^2 - \tfrac{1}{3}({\xi_1}^4 + {\eta_1}^4)\right) + \cdots\right) + \sin 4\lambda\left(\tfrac{4}{3}L^{-2}({\xi_1}^3\eta_1 - {\eta_1}^3\xi_1) + \cdots\right) + \cdots\end{aligned}$$
- $$\begin{aligned}\frac{a}{r} = {} & 1 + \cos\lambda\left(L^{-\frac{1}{2}}\xi_1 - \tfrac{1}{4}L^{-\frac{3}{2}}({\eta_1}^2\xi_1 + {\xi_1}^3) + \cdots\right) + \sin\lambda\left(-L^{-\frac{1}{2}}\eta_1 + \tfrac{1}{4}L^{-\frac{3}{2}}({\xi_1}^2\eta_1 + {\eta_1}^3) + \cdots\right) + \\ & + \cos 2\lambda\left(L^{-1}({\xi_1}^2 - {\eta_1}^2) + \tfrac{7}{12}L^{-2}({\eta_1}^4 - {\xi_1}^4) + \cdots\right) + \sin 2\lambda\left(-2L^{-1}\xi_1\eta_1 + \tfrac{7}{6}L^{-2}({\eta_1}^3\xi_1 + \eta_1{\xi_1}^3) + \cdots\right) + \\ & + \cos 3\lambda\left(\tfrac{9}{8}L^{-\frac{3}{2}}({\xi_1}^3 - 3{\eta_1}^2\xi_1) + \cdots\right) + \sin 3\lambda\left(\tfrac{9}{8}L^{-\frac{3}{2}}({\eta_1}^3 - 3{\xi_1}^2\eta_1) + \cdots\right) + \\ & + \cos 4\lambda\left(L^{-2}\left(-8{\xi_1}^2{\eta_1}^2 + \tfrac{4}{3}({\xi_1}^4 + {\eta_1}^4)\right) + \cdots\right) + \sin 4\lambda\left(\tfrac{16}{3}L^{-2}\left({\eta_1}^3\xi_1 - {\xi_1}^3\eta_1\right) + \cdots\right) + \cdots\end{aligned}$$

RESULTS

Precision of disturbing function approximation

- Expansion was made to 1st degree of small parameter and to 6th degree of eccentric and oblique Poincare elements.
- For example, precision of the approximation was calculated for Solar system and 47 Uma star system Kepler orbital elements:

	Jupiter	Saturn	Uranus	Neptune
a, AU	5.203	9.537	19.189	30.070
e	0.0484	0.0539	0.0473	0.0086
i, *rad*	0.0228	0.0434	0.0135	0.0309
Ω, *rad*	1.754	1.983	1.291	2.300
ω, *rad*	4.801	5.865	1.685	4.636
l, *rad*	0.328	5.591	2.495	4.673

Kepler elements for Solar system w.r.t the epoch J2000.0 and correspond to mean ecliptic

47 Uma b	47 Uma c	47 Uma d
2.10	3.60	11.6
0.032	0.098	0.16
0.10	0.10	0.10
3.76	5.36	1.48
5.83	5.14	1.92
0.89	1.70	1.92

red items from www.exoplanet.eu
black items have arbitrary values

Precision of disturbing function approximation

Solar system

Major part of disturbing function:

$$\frac{m_1 m_2}{|\vec{\rho}_1-\vec{\rho}_2|}+\frac{m_1 m_3}{|\vec{\rho}_1-\vec{\rho}_3|}+\frac{m_1 m_4}{|\vec{\rho}_1-\vec{\rho}_4|}+\frac{m_2 m_3}{|\vec{\rho}_2-\vec{\rho}_3|}+\frac{m_2 m_4}{|\vec{\rho}_2-\vec{\rho}_4|}+\frac{m_3 m_4}{|\vec{\rho}_3-\vec{\rho}_4|}$$

i	j	accurate formula	series expansion	relative differences
1	2	$6.247339 \cdot 10^{-2}$	$6.247214 \cdot 10^{-2}$	$2 \cdot 10^{-5}$
1	3	$2.117572 \cdot 10^{-3}$	$2.117594 \cdot 10^{-3}$	$1 \cdot 10^{-5}$
1	4	$1.598993 \cdot 10^{-3}$	$1.598990 \cdot 10^{-3}$	$2 \cdot 10^{-6}$
2	3	$5.715344 \cdot 10^{-4}$	$5.715304 \cdot 10^{-4}$	$7 \cdot 10^{-6}$
2	4	$4.429494 \cdot 10^{-4}$	$4.429499 \cdot 10^{-4}$	$1 \cdot 10^{-6}$
3	4	$1.953597 \cdot 10^{-4}$	$1.953599 \cdot 10^{-4}$	$1 \cdot 10^{-6}$
Σ		$6.932077 \cdot 10^{-2}$	$6.932205 \cdot 10^{-2}$	$2 \cdot 10^{-5}$

Precision of disturbing function approximation

Solar system

Second part of disturbing function:

$$\frac{m_2\left(2\vec{r}_2\vec{R}_2+\mu\vec{R}_2{}^2\right)}{r_2\tilde{R}_2(r_2+\tilde{R}_2)}+\frac{m_3\left(2\vec{r}_3\vec{R}_3+\mu\vec{R}_3{}^2\right)}{r_3\tilde{R}_3(r_3+\tilde{R}_3)}+\frac{m_4\left(2\vec{r}_4\vec{R}_4+\mu\vec{R}_4{}^2\right)}{r_4\tilde{R}_4(r_4+\tilde{R}_4)}$$

i	accurate formula	series expansion	relative differences
1	$1.583793\cdot10^{-2}$	$1.583794\cdot10^{-2}$	$4\cdot10^{-7}$
2	$9.514375\cdot10^{-5}$	$9.514331\cdot10^{-5}$	$5\cdot10^{-6}$
3	$6.555984\cdot10^{-6}$	$6.555954\cdot10^{-6}$	$5\cdot10^{-6}$
Σ	$1.593963\cdot10^{-2}$	$1.593964\cdot10^{-2}$	$3\cdot10^{-7}$

Whole disturbing function:

accurate formula	series expansion	relative differences
$8.526040\cdot10^{-2}$	$8.526169\cdot10^{-2}$	$2\cdot10^{-5}$

Precision of disturbing function approximation

47 Uma
star system

Major part of disturbing function:

$$\frac{m_1 m_2}{|\vec{\rho}_1 - \vec{\rho}_2|} + \frac{m_1 m_3}{|\vec{\rho}_1 - \vec{\rho}_3|} + \frac{m_2 m_3}{|\vec{\rho}_2 - \vec{\rho}_3|}$$

i	*j*	accurate formula	series expansion	relative differences
1	2	0.265913	0.265903	$4 \cdot 10^{-5}$
1	3	0.310104	0.310087	$5 \cdot 10^{-5}$
2	3	0.084960	0.084985	$2 \cdot 10^{-5}$
Σ		0.660976	0.660949	$4 \cdot 10^{-5}$

Precision of disturbing function approximation

47 Uma
star system

Second part of disturbing function:

$$\frac{m_2\left(2\vec{r}_2\vec{R}_2+\mu\vec{R}_2^{\,2}\right)}{r_2\tilde{R}_2(r_2+\tilde{R}_2)}+\frac{m_3\left(2\vec{r}_3\vec{R}_3+\mu\vec{R}_3^{\,2}\right)}{r_3\tilde{R}_3(r_3+\tilde{R}_3)}$$

i	accurate formula	series expansion	relative differences
1	0.0496716	0.0496752	$7 \cdot 10^{-5}$
2	0.0254878	0.0254881	$1 \cdot 10^{-5}$
Σ	0.0751594	0.0751633	$5 \cdot 10^{-5}$

Whole disturbing function:

accurate formula	series expansion	relative differences
0.6367925	0.6367620	$5 \cdot 10^{-5}$

CONCLUSION

- We have got expansion of the Hamiltonian for planetary system with 4 planets into the Poisson series in all elements.
- The expansion was made to 6th degree of orbital elements and to 1st degree of small parameter.
- Legendre polynomials were saved as symbols.
- Estimation accuracy of disturbing function (h_1) expansion is presented. Relative difference between series estimation and accurate formula is about $10^{-4} – 10^{-5}$.
- The Hamiltonian is $h_0 + \mu h_1$.
- So, Poisson series for the Hamiltonian was constructed with precision about $10^{-7} – 10^{-8}$.
- Now we are constructing the expansion for the Hamiltonian to 11th degree of orbital elements and 2nd degree of small parameter.

THANK YOU
FOR ATTENTION