THE PROBABILISTIC APPROACH TO THE DESCRIPTION OF THE CHANDLER WOBBLE

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INTRODUCTION. The probabilistic approach to the description of the Chandler wobble (CW) was proposed by Arato, Kolmogorov and Sinai in [Arato et al., 1962]. They assumed that the moment of the forces causing the CW is a stationary random process with correlation time τ_{cor} which is small in comparison with length of the row of observations. Then, the CW itself can be regarded as a diffusion Markovian process with discrete time; wherein the sampling step Δ must satisfy the condition: $\Delta >> \tau_{cor}$.

There was shown in [Tsurkis et al. 2009], that the probabilistic model does not contradict with observations. Besides, evaluations for τ_{cor} and the diffusion coefficient d were obtained:

$$\tau_{cor} < 100\,days, \qquad d = 1.1\cdot 10^{-16} .....1.8\cdot 10^{-16}\ rad^2/day. \qquad (1)$$

An equally important task is the studying of processes causing CW. Polar motion is due to several reasons, the main of which, apparently, is the impact of ocean and atmosphere to the solid Earth [Gross et al, 2003], [Barnes et al., 1983]. The article [Tsurkis et al., 2012b] is devoted to analysis of the data on ocean angular momentum. This report is based on the results obtained there.

1. Data

The series $\chi_k(t), k = 1,2$ of the ocean excitation functions of the International Earth Rotation and Reference System Service for 1982-2003 (IERS, http://www.iers.org) are analyzed. These data refer to a right rectangular coordinate system, where axes x_1 and x_2 are lying in the equatorial plane, and x_1 is directed along projection of the Greenwich Meridian to this plane. Components M_1 and M_2 of the torque exerted by the ocean, are

$$M_1 = \omega C\left(\Omega\chi_2 - \dot{\chi}_1\right), \quad M_2 = -\omega C\left(\Omega\chi_1 + \dot{\chi}_2\right),$$

where $\Omega = 2\pi/day$ is the average frequency of the Earth's rotation; $C = 7.04\times 10^{37}\,\mathrm{kg\cdot m^2}$ is the equatorial moment of inertia of the Earth; $\omega \approx 0{,}0145\,day^{-1}$ is the frequency of free nutation (Chandler frequency). The procedure for calculating the excitation functions $\chi_1(t)$ and $\chi_2(t)$ is described in [Gross et al, 2003].

2. Statement of the Problem

The CW is described by the linearized Liouville equation expressing the law of angular momentum conservation [Munk, MacDonald, 1964].

$$\frac{d}{dt}x_1 + \frac{1}{2Q}\frac{d}{dt}x_2 + \omega x_2 = f_1, \qquad \frac{d}{dt}x_2 - \frac{1}{2Q}\frac{d}{dt}x_1 - \omega x_1 = f_2, \qquad (2)$$

here $x_k, k = 1,2$ are dimensionless coordinates of the pole; Q is mantle quality factor (at frequencies of the order of ω); $f_k = M_k/(\Omega C)$. Within the framework of the probabilistic approach f_k are random functions of time. The pair f_1, f_2 will be referred to as a random load. If nothing but the ocean impacts the pole motion, then

$$f_1 = \omega\left(\chi_2 - \frac{\dot{\chi}_1}{\Omega}\right) \qquad f_2 = -\omega\left(\chi_1 + \frac{\dot{\chi}_2}{\Omega}\right), \qquad (3)$$

where $\chi_k(t)$ are the ocean excitation functions. Obviously, the right-hand side of (2) contains law-frequency, year and half-year modes. However, we will mean below f_k as random parts of functions (3), so $\mathrm{M}f_1 = \mathrm{M}f_2 = 0$. The hypothesis to be checked is as follows: the load $(f_1(t), f_2(t))$ is a normal stationary stochastic process with small time of correlation. A mathematical model of such process is a two-dimensional white noise: we assume that

$$\mathrm{M}\left(f_1(t_1)f_1(t_2)\right) = F_{11}\,\delta(t_2 - t_1), \quad \mathrm{M}\left(f_2(t_1)f_2(t_2)\right) = F_{22}\,\delta(t_2 - t_1)$$
$$\mathrm{M}\left(f_1(t_1)f_2(t_2)\right) = F_{12}\,\delta(t_2 - t_1), \qquad (4)$$

where F_{11}, F_{22}, F_{12} are components of a non-negative symmetric matrix, which we call the diffusion matrix and denote by $\mathbf{F}$:

$$\mathbf{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix}.$$

The aim of this work is to test the statistical hypothesis (4). Along the way, we shall evaluate the correlation time τ_{cor} and parameters characterizing the matrix $\mathbf{F}$, namely the diffusion coefficient a

$$a(\mathbf{F}) = \mathrm{Tr}\,\mathbf{F} = F_{11} + F_{22} \qquad (5)$$

and anisotropy constant κ

$$\kappa(\mathbf{F}) = 1 - \frac{F_2}{F_1} = \frac{2\sqrt{(F_{11} - F_{22})^2 + 4F_{12}^2}}{F_{11} + F_{22} + \sqrt{(F_{11} - F_{22})^2 + 4F_{12}^2}}, \qquad (6)$$

here F_1 and $F_2 \le F_1$ eigenvalues of the matrix $\mathbf{F}$. We disclaim the assumption that the ocean load is isotropic, i.e. that $F_2 = F_1$.

3. Course of the solution. Main results

Let us consider the equation (2) without dissipation (i.e. in case $Q = \infty$). It can be written in the form:

$$\dot{x} - i\omega x = f(t),$$

where $x = x_1 + ix_2$, $f = f_1 + if_2$. If $x(0) = 0$, then

$$x(t) = \int_0^t f(\tau)\exp i\omega(t - \tau)\,d\tau$$

In terms of the excitation functions,

$$x = -\frac{\omega}{\Omega}\left(\chi(t) - \chi(0)\exp i\omega t\right) + \left(1 + \frac{\omega}{\Omega}\right)i\int_0^t \chi(\tau)\exp i\omega(t - \tau)d\tau, \qquad (7)$$

where $\chi = \chi_1 + i\chi_2$, see Fig. 1.

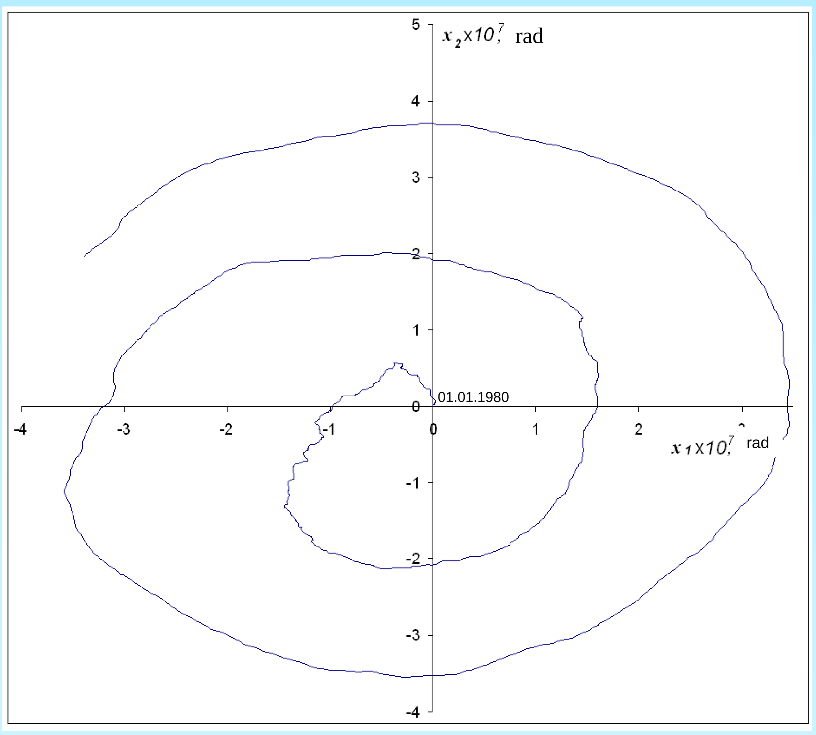

Fig.1. The function $x(t)$ calculated by formula (7) for 1000 days. The moment $t = 0$ corresponds to 1.01.1980 г.

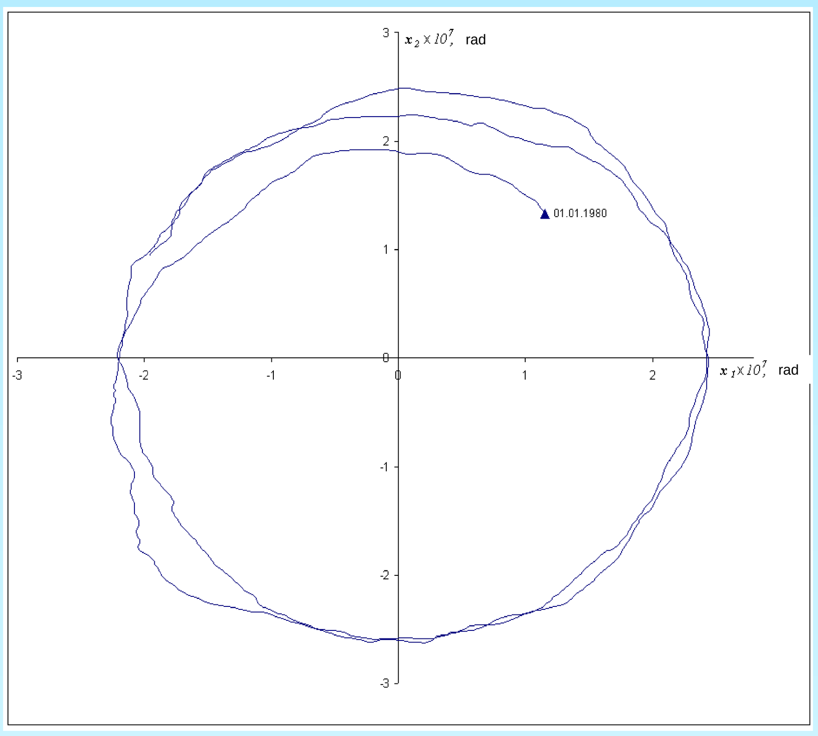

Fig.2. Ocean component of CW for 1000 days (from 01.01.1980)

If the hypothesis (4) is true, then the function $x(t)$ is a realization of a Markovian process in virtue of Doob's theorem (see, for example, [Tikhonov et al., 1977]). Next, we must eliminate the deterministic part of function (7) consisting of low-frequency component, the mode with a period of 1 year and its first overtone. This is the problem of independent interest; a variant of spectral analysis proposed in [Tsurkis et al., 2012a] allows one to do it correctly. Below, we denote as x(t) the random part of the function (7), see. Fig. 2; it will be referred to as the ocean component of CW.

Let us consider the function $x_{u,v}(t)$:

$$x_{u,v} = u\,x(t) + v\left(\overline{x}(t) + 2i\omega\int_0^t \overline{x(\tau)}\exp i\omega(t - \tau)\,d\tau\right),$$

which is the solution of the differential equation

$$\dot{x}_{u,v} - i\omega x_{u,v} = uf(t) + v\,\overline{f(t)} \qquad (8)$$

If conditions (4) are satisfied, $x_{u,v}(t)$ is a Markovian process too. It turns out that the elements of the matrix can be estimated by formulas

$$F_{11} = \langle x_{1/2,1/2}, x_{1/2,1/2}\rangle \qquad F_{22} = \langle x_{1/2,-1/2}, x_{1/2,-1/2}\rangle \qquad F_{12} = \frac{1}{8}\left(\langle x_{1,i}, x_{1,i}\rangle - \langle x_{i,1}, x_{i,1}\rangle\right), \qquad (9)$$

where $\langle *,*\rangle$ is Wiener-Liouville scalar product, which was taken into account in [Tsurkis et al., 2012a]:

$$\langle z, w\rangle = \frac{1}{T}\sum_{n=0}^{N-1}\left(z((n+1)\Delta) - e^{i\omega\Delta}z(n\Delta)\right)\left(\overline{w((n+1)\Delta)} - e^{-i\omega\Delta}\,\overline{w(n\Delta)}\right)$$

here $T = N\Delta$ is length of the series of observations. If the main hypothesis (4) is true, the estimates (9) are consistent and nonshifted; proof of this fact uses statistical independence of increments $x_{u,v}((n+1)\Delta) - e^{i\omega\Delta}x_{u,v}(n\Delta)$, which presents the sequence of absence of dissipative term in the left-hand side of (8). Starting with some value of the sampling parameter Δ_0, estimations (9) should not depend on Δ. And, indeed, functions $F_{11}(\Delta)$ and $F_{22}(\Delta)$ increase with $\Delta < \Delta_0 = 100\,days$; then, if $\Delta > \Delta_0$, they range near the values:

$$F_{11}(\Delta_0) \approx 1.00\cdot 10^{-17}\,rad^2/day, F_{22}(\Delta_0) \approx 0.57\cdot 10^{-17}\,rad^2/day, \qquad (10)$$

Fig. 3a, b. For the function $F_{12}(\Delta)$ the "threshold" value is $\Delta \approx 50\,days$; this function decreases from 0 to $-0.15\cdot 10^{-17}$ rad^2/day In the interval 0…50 days; if $\Delta > 50\,days$ it remains close to value

$$F_{12} \approx -0.11\cdot 10^{-17}\,rad^2/day, \qquad (11)$$

and also performs oscillations whose amplitude increases with increasing of sampling parameter, Fig. 3c. *So, results presented in Figure 3 are consistent with the basic hypothesis;* for the correlation time we find:

$$\tau_{cor} < \Delta_0 = 100\,days.$$

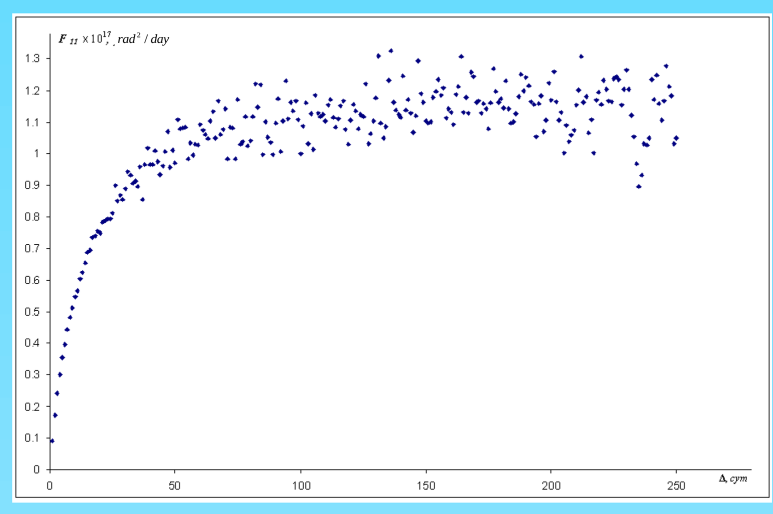

Fig 3. a).

The coefficient of diffusion a and the anisotropy constant κ one can roughly estimate, using (10), (11) and formulas (5) and (6):

$$a \approx 1.6\cdot 10^{-17}\,rad^2/day, \quad \kappa \approx 0.47$$

To find confidence intervals for a and κ we must use the fact that estimations (9) for $N >> 1$ are normally distributed values (it is the sequence of (4)). We have to find a statistical relationship between these estimates, to build a three-dimensional "confidence region" Φ_P for matrix $\mathbf{F} \equiv (F_{11}, F_{12}, F_{22})$ and probability P; and then to find the marginal values of functions $a(\mathbf{F})$ and $\kappa(\mathbf{F})$ on the set Φ_P, using (5) and (6): marginal values will be the required boundaries of the confidence intervals. Thus, we must consider the matrix $\mathbf{F}$ on the whole as three-dimensional object.

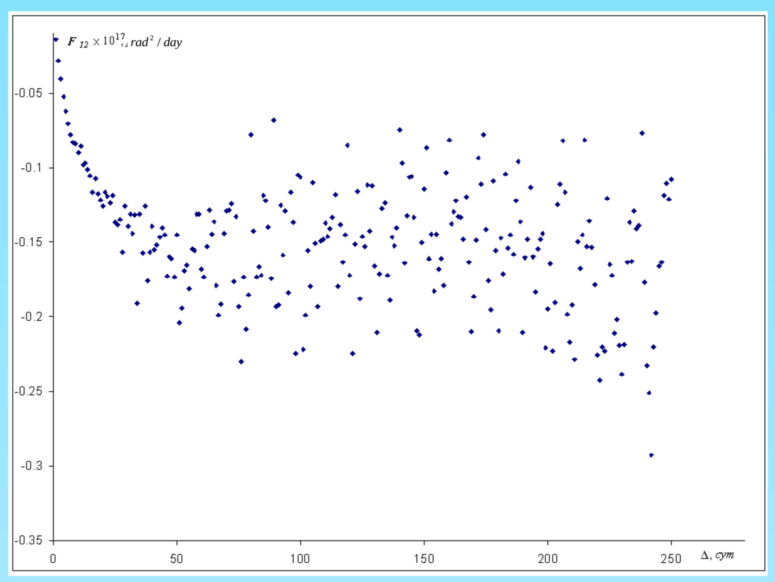

Fig 3. b).

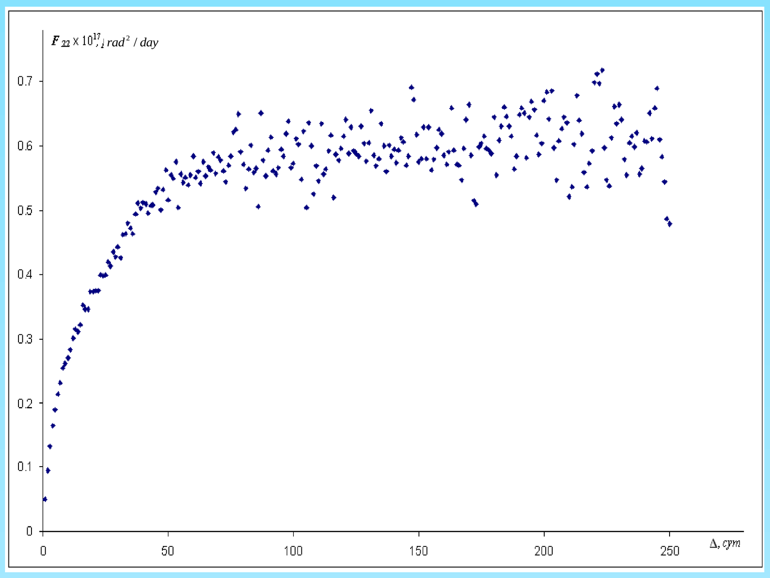

Fig 3. c).

FIG. 3. Estimates (8) as functions of sampling step: *a*) - $F_{11}(\Delta)$, b) - $F_{22}(\Delta)$, *c*) - $F_{12}(\Delta)$

The plan invented gives the following result: parameters a and κ belong to intervals:

$$a = 1.3\cdot 10^{-17} .....2.2\cdot 10^{-17}\ rad^2/day, \quad \kappa = 0.06...0.65. \qquad (12)$$

with probability $P > 0.92$. The confidential interval for the anisotropy constant lies entirely in the positive area; therefore we must consider the ocean load acting to the solid Earth as anisotropic.

4. The contribution of the ocean to the CW excitation

The diffusion coefficient $a = \mathrm{Tr}\,\mathbf{F}$ is the main characteristic of the random load. Indeed, if $\sin(\omega\Delta)/\omega\Delta << 1$ and $Q >> 1$, then regardless of the relationship between the eigenvalues of the matrix $\mathbf{F}$, the sequence $x(n\Delta), n = 1,2,...$ can be interpreted as a discrete-time process with isotropic diffusion matrix, the trace of which equals to a [Tsurkis et al., 2014]. But in the case of an isotropic load the CW amplitude $A(t) = \sqrt{x_1^2 + x_2^2}$ is also a Markovian process. If $Q < \infty$ this process can be characterized by stationary amplitude, i.e. the expectation $A_{st} = \mathrm{M}A(t)$, which is proportional $\sqrt{a}$ [Tsurkis et al., 2011].

Therefore, the ocean share of CW can be estimated by value $\sqrt{a/d}$, where d is "general" diffusion coefficient characterizing the pole motion as a whole. Comparing (1) and (12), we see that $a \sim 0.1\,d$. This means that in absence of other sources of excitation, the average amplitude of CW would be approximately one-third of the real one. On the other hand, if we subtract the ocean torque from general angular moment acting on CW, the expectation of CW amplitude decreases slightly, by approximately 5%. It is explained by non-linear dependence of average amplitude from $a = \mathrm{Tr}\,\mathbf{F}$.

5. Conclusions

It is shown that the data on the torque exerted by the ocean on the solid Earth do not contradict the statistical hypothesis. Namely, it can be interpreted as an anisotropic stationary random process with a time of autocorrelation $\tau_{cor} < 100\,days$. In context of the probabilistic approach, an estimate of the contribution of the ocean to the CW was obtained: the one-third part of the CW amplitude can be explained by ocean.

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