Анализ поля скоростей звезд тонкого диска по данным Gaia DR2 и PMA с помощью зонных векторных сферических функций

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Gaia DR2 content

	# sources in Gaia DR2	# sources in Gaia DR1
Total number of sources	**1,692,919,135**	**1,142,679,769**
Number of 5-parameter sources	1,331,909,727	2,057,050
Number of 2-parameter sources	361,009,408	1,140,622,719
Sources with mean G magnitude	1,692,919,135	1,142,679,769
Sources with mean G_{BP}-band photometry	1,381,964,755	-
Sources with mean G_{RP}-band photometry	1,383,551,713	-
Sources with radial velocities	7,224,631	-
Variable sources	550,737	3,194
Known asteroids with epoch data	14,099	-
Gaia-CRF sources	556,869	2,191
Effective temperatures (T_{eff})	161,497,595	-
Extinction (A_G) and reddening ($E(G_{BP}-G_{RP})$)	87,733,672	-
Sources with radius and luminosity	76,956,778	-

FGK stars

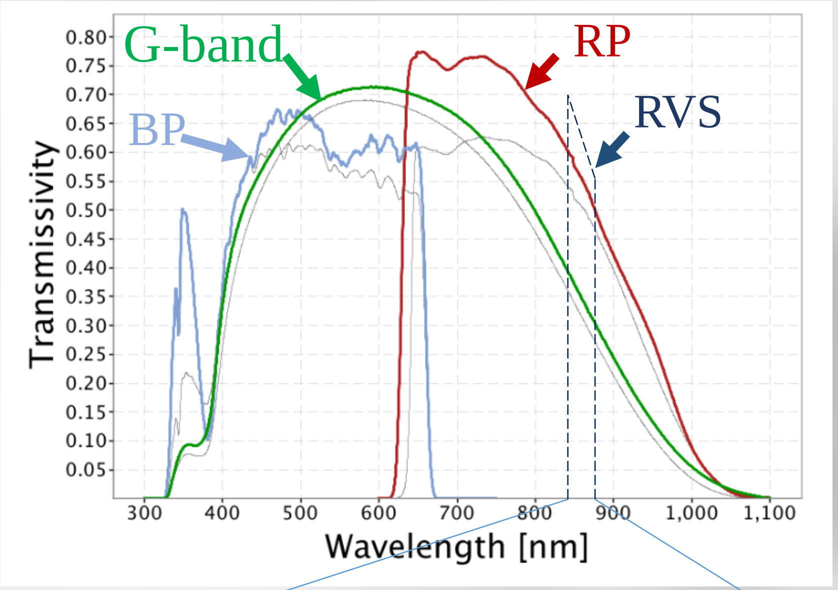
https://www.cosmos.esa.int/web/gaia/dr2

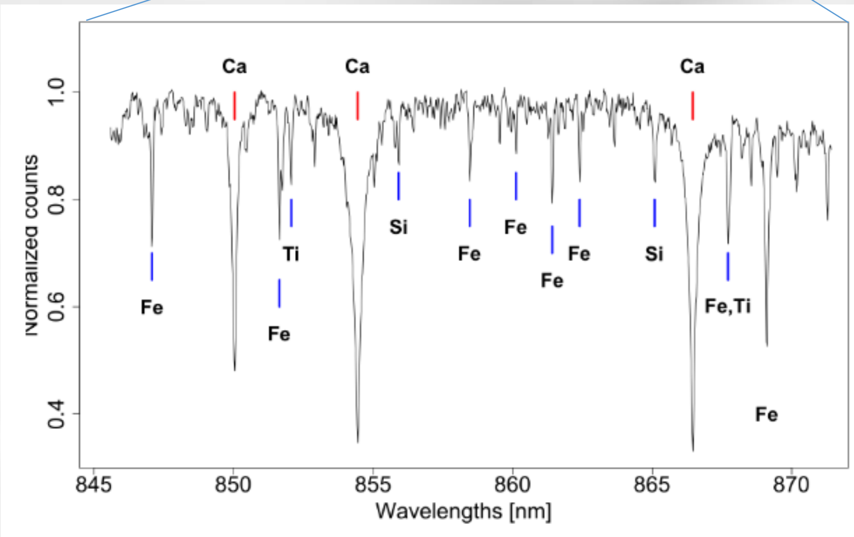
Data used in calculations

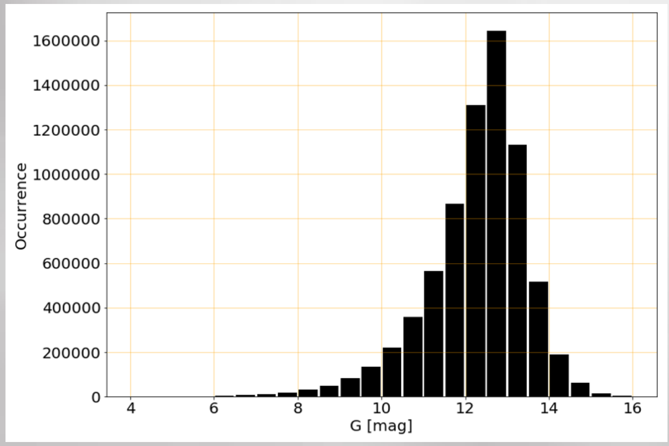
7.2 mln **Gaia DR2** FGK stars with measured radial velocities

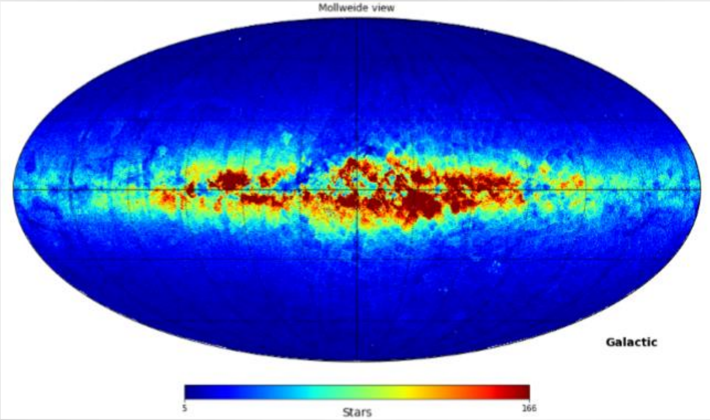
Properties of the Gaia DR2 data (Brown & Gaia co, A&A, 2018):

Data	**Typical uncertainty**
α, δ, parallaxes	0.02-0.04 mas at G < 15
$\mu_\alpha \cos\delta$, μ_δ	0.07 mas/yr at G < 15
radial velocities	< 0.1 km/s at G_{RVS} < 9, 0.5 km/s at G_{RVS} = 11.75
G magnitude	0.3 mmag at G < 13 2 mmag at G = 17









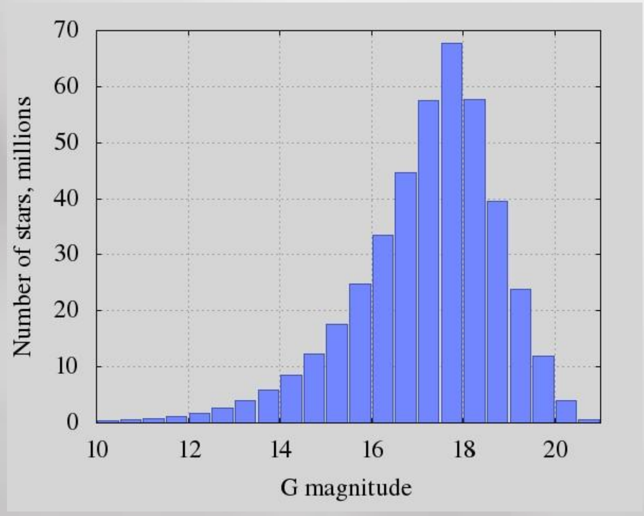
(Katz, D. et al, arXiv:1804.09372v1, 2018)

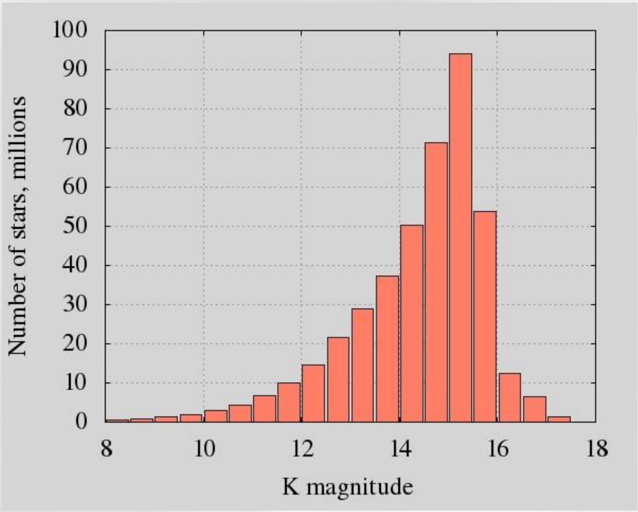
Data used in calculations

PMA catalogue, ~**421 mln** objects

Data	Typical uncertainty
α, δ	10 mas
$\mu_\alpha \cos\delta$, μ_δ	2-5 mas/yr at $10 < G < 17$

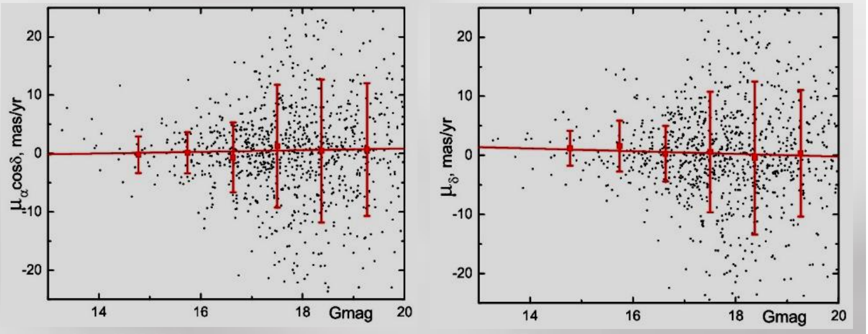
The system of PMA proper motions is **independent** on ICRF and HCRF, and together with its own positions in the range from 14 to 21 magnitude represents an **independent realization** of a quasi-inertial reference frame in the optical and near infrared wavelength ranges.



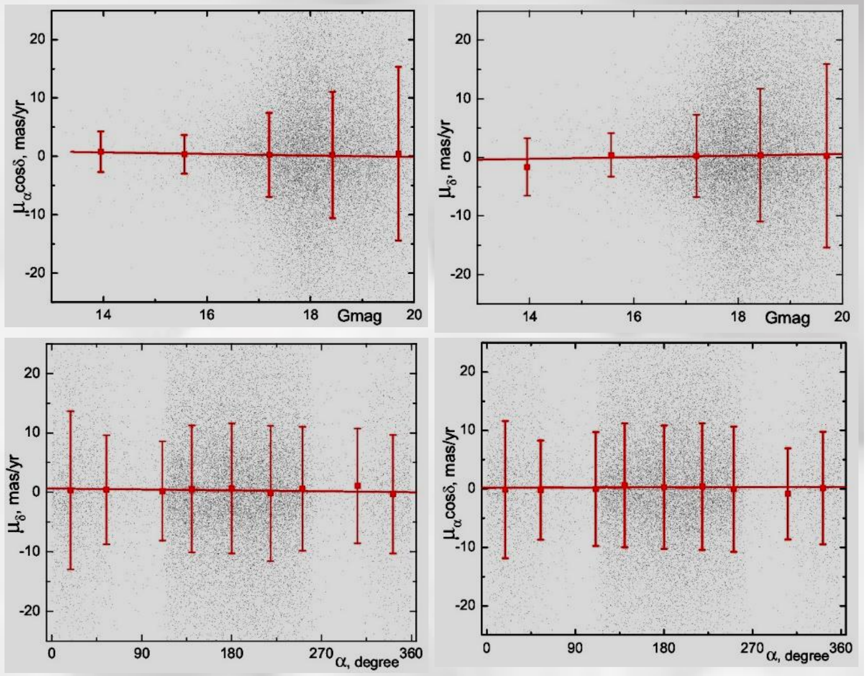


(Akhmetov, Fedorov, Velichko, Shulga, MNRAS, 2017 vol. 469, 763A)

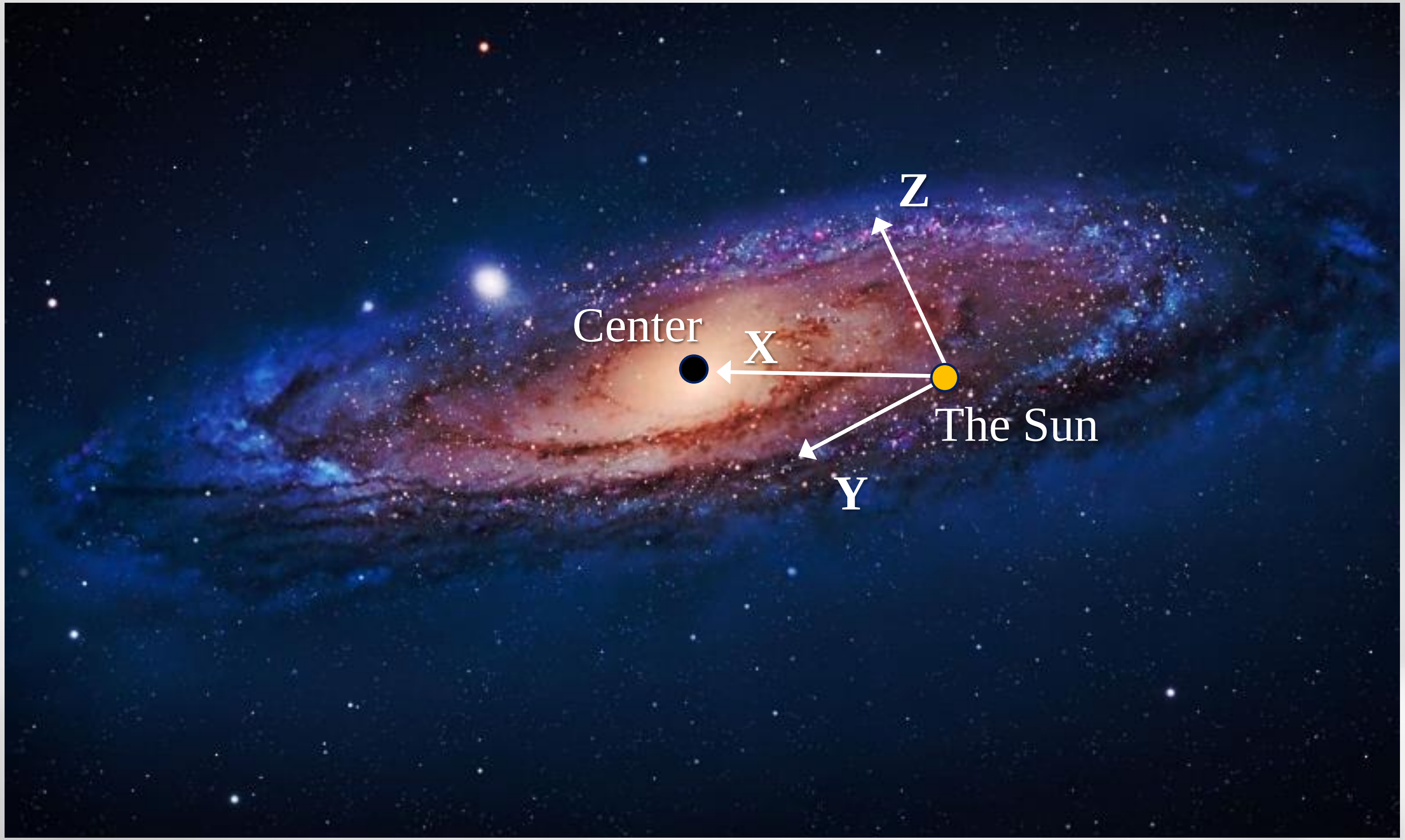
Scatter of individual proper motions for ICRF quasars



Scatter of individual proper motions for LQAC3 quasars



The Galactic coordinate system



X – to the Galactic center
Y – to the direction of the Galactic rotation
Z – perpendicular to the Galactic plane

Selection of thin disk stars

Velocities of stars in the Galactic coordinate system

$$
\begin{aligned}
U &= V_r \cos l \cos b - \eta\, r\, \mu_l \sin l \cos b - \eta\, r\, \mu_b \cos l \sin b \\
V &= V_r \sin l \cos b + \eta\, r\, \mu_l \cos l \cos b - \eta\, r\, \mu_b \sin l \sin b \\
W &= V_r \sin b + \eta\, r\, \mu_b \cos b
\end{aligned}
$$

Reducing to the LSR

$$(U, V, V)_{\mathrm{res}} = (U - U_\odot, V - V_\odot, W - W_\odot)$$

where

$$(U, V, W)_\odot = (11.1^{+0.69}_{-0.75}, 12.24^{+0.47}_{-0.47}, 7.25^{+0.37}_{-0.36})\ \mathrm{km\,s^{-1}}$$

[Schönrich, R., Binney, J., Dehnen, W., MNRAS, 403, 1829-1833, 2010]

The Bensby's kinematic method

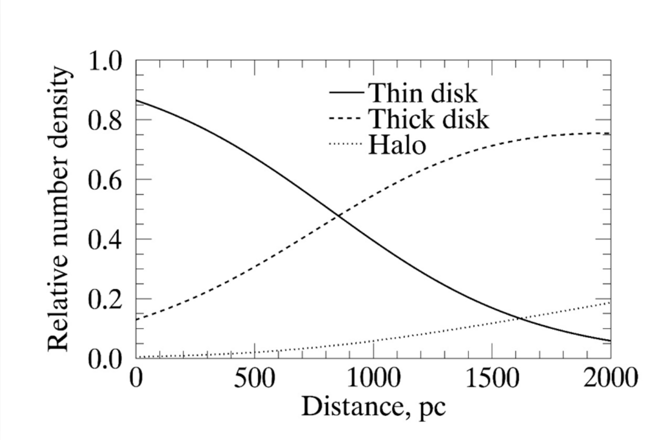
[T. Bensby, S. Feltzing, I. Lundström, A&A 410, 527-551 (2003)]

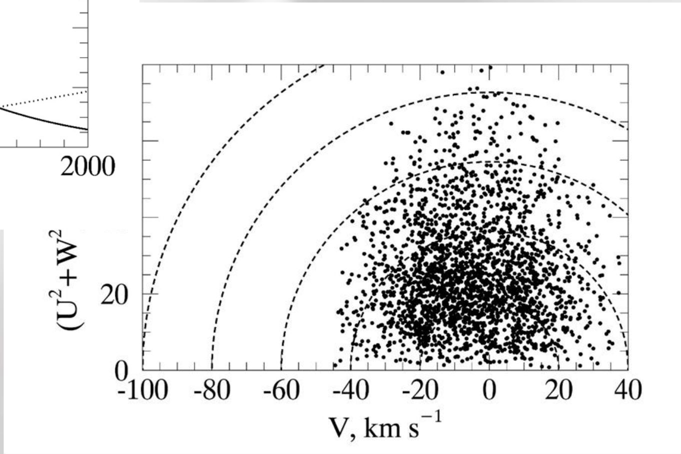
$$f(U, V, W) = k \cdot \exp\left(-\frac{U_{\mathrm{LSR}}^2}{2\sigma_U^2} - \frac{(V_{\mathrm{LSR}} - V_{\mathrm{asym}})^2}{2\sigma_V^2} - \frac{W_{\mathrm{LSR}}^2}{2\sigma_W^2}\right)$$

$$k = \frac{1}{(2\pi)^{3/2}\sigma_U \sigma_V \sigma_W}$$

$$\mathrm{TD/D} = \frac{X_{\mathrm{TD}}}{X_{\mathrm{D}}} \cdot \frac{f_{\mathrm{TD}}}{f_{\mathrm{D}}}$$

	X	σ_U	σ_V	σ_W	V_{asym}
			[km s^{-1}]		
Thin disk (D)	0.94	35	20	16	–15
Thick disk (TD)	0.06	67	38	35	–46
Halo (H)	0.0015	160	90	90	–220

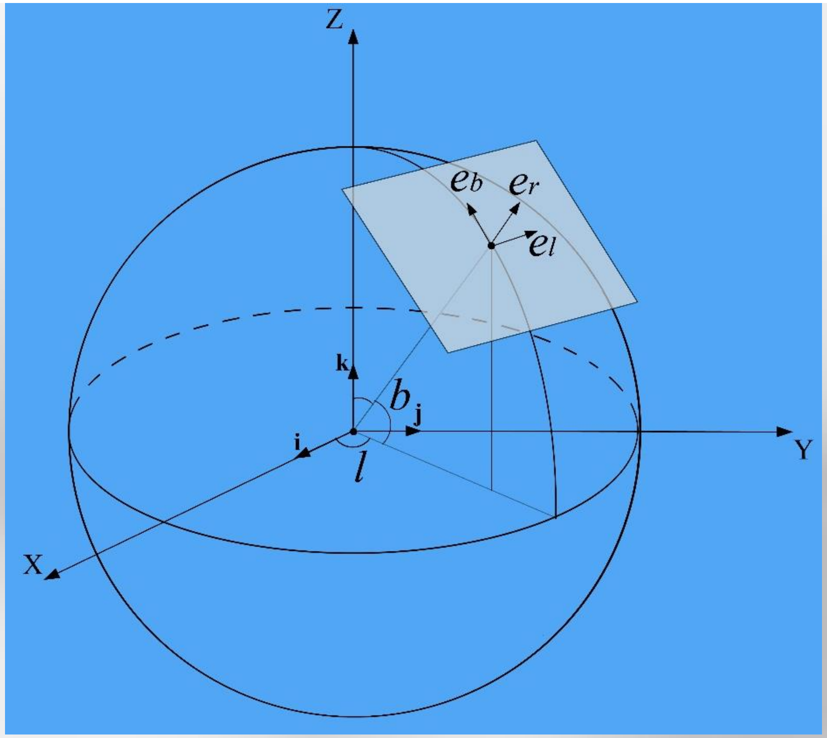




D/TD > 2 + Z = ±300 pc → ~3.3 mln stars

Stellar velocity field

$$\mathbf{U}(l,b) = V_r/r\mathbf{e}_r + \kappa\mu_l cos\, b\mathbf{e}_l + \kappa\mu_b\mathbf{e}_b$$



V_r - radial velocity

$\mu_l,\ \mu_b$ - proper motion components

$\kappa = 4.738$ - converting factor from mas/yr to km/s kpc^{-1}

The Ogorodnikov - Milne model

(Ogorodnikov, 1965)

$$\mathbf{U}(l,b) = \mathbf{V}_0 + \Omega \times \mathbf{r} + \mathbf{M}^+ \times \mathbf{r}$$

$\mathbf{V}_0 : U, V, W$

(Solar motion)

$\Omega : \omega_1, \omega_2, \omega_3$

(Rigid-body rotation)

(Deformation in planes)

$M_{11}^+ \ M_{22}^+ \ M_{33}^+$

(Contraction - expansion)

$M_{12}^+,\ M_{13}^+,\ M_{23}^+$

If radial velocities are not used, only linear combination can be derived

$$M_{11}^* = M_{11}^+ - M_{22}^+$$

$$M_{33}^* = M_{33}^+ - M_{22}^+$$

The OMM equations in the Galactic coordinate system

3D

$$\eta\,\mu_l\cos b = X_\odot/r\sin l - Y_\odot/r\cos l - \omega_1\sin b\cos l - \omega_2\sin b\sin l + \omega_3\cos b + M^+_{12}\cos b\cos 2l - $$
$$-M^+_{13}\sin b\sin l + M^+_{23}\sin b\cos l - 0.5\,M^+_{11}\cos b\sin 2l + 0.5\,M^+_{22}\cos b\sin 2l$$

$$\eta\,\mu_b = X_\odot/r\cos l\sin b + Y_\odot/r\sin l\sin b - Z_\odot/r\cos b + \omega_1\sin l - \omega_2\cos l - 0.5\,M^+_{12}\sin 2b\sin 2l + $$
$$+M^+_{13}\cos 2b\cos l + M^+_{23}\cos 2b\sin l - 0.5\,M^+_{11}\sin 2b\cos^2 l - 0.5\,M^+_{22}\sin 2b\sin^2 l + 0.5\,M^+_{33}\sin 2b$$

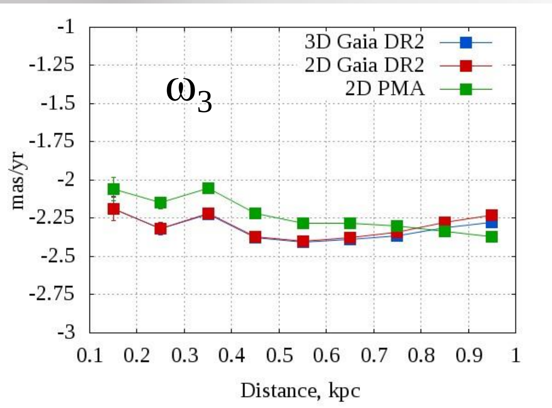
$$V_r/r = -X_\odot/r\cos l\cos b - Y_\odot/r\sin l\cos b - Z_\odot/r\sin b + M^+_{13}\sin 2b\cos l + M^+_{23}\sin 2b\sin l + $$
$$+M^+_{12}\cos^2 b\sin 2l + M^+_{11}\cos^2 b\cos^2 l + M^+_{22}\cos^2 b\sin^2 l + M^+_{33}\sin^2 b$$

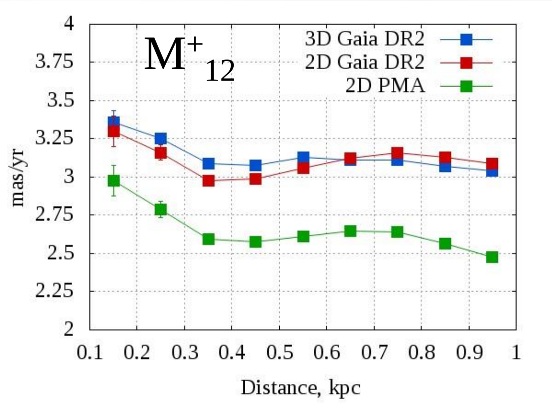
2D

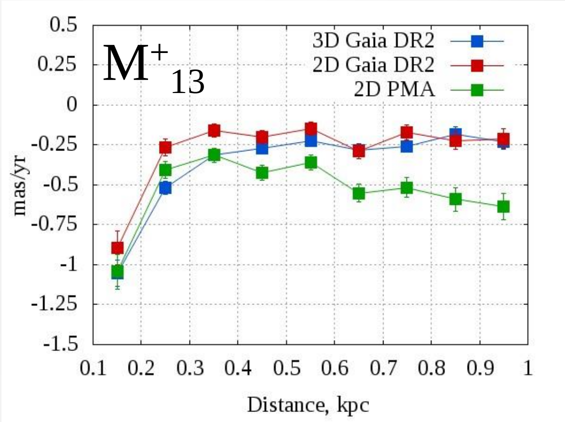
$$\eta\,\mu_l\cos b = X_\odot/r\sin l - Y_\odot/r\cos l - \omega_1\sin b\cos l - \omega_2\sin b\sin l + \omega_3\cos b + M^+_{12}\cos b\cos 2l - $$
$$-M^+_{13}\sin b\sin l + M^+_{23}\sin b\cos l - 0.5\,(M^+_{11} - M^+_{22})\cos b\sin 2l$$

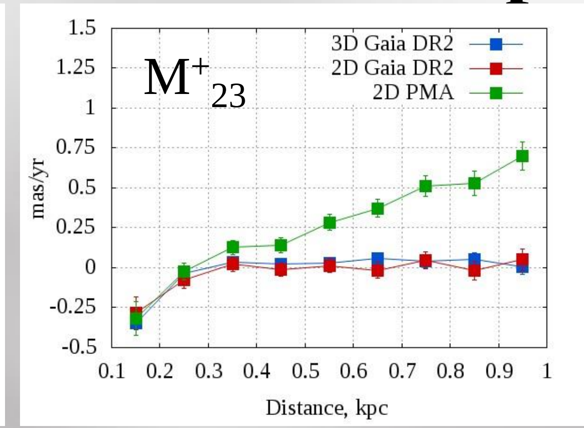
$$\eta\,\mu_b = X_\odot/r\cos l\sin b + Y_\odot/r\sin l\sin b - Z_\odot/r\cos b + \omega_1\sin l - \omega_2\cos l - 0.5\,M^+_{12}\sin 2b\sin 2l + $$
$$+M^+_{13}\cos 2b\cos l + M^+_{23}\cos 2b\sin l - 0.5\,(M^+_{11} - M^+_{22})\sin 2b\cos^2 l + 0.5\,(M^+_{33} - M^+_{22})\sin 2b$$

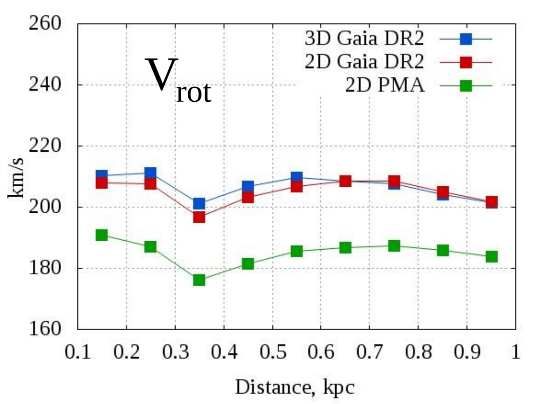
The OMM parameters depending on distance to the stellar sample

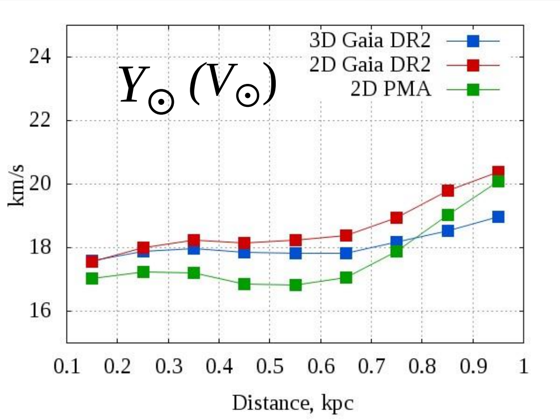


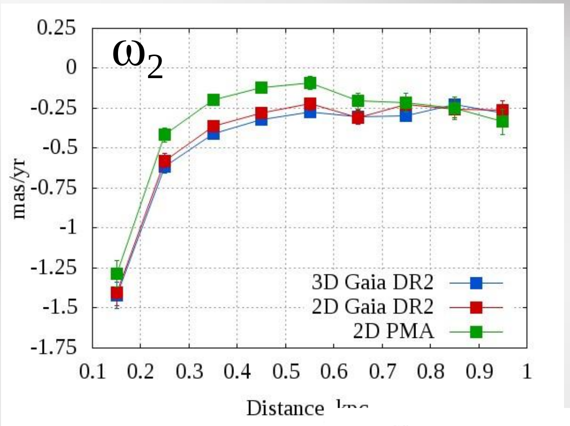


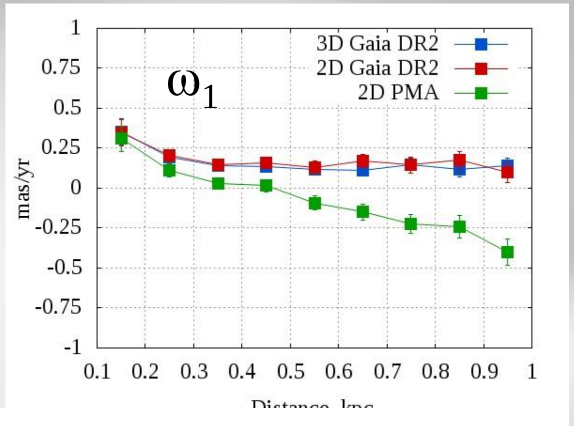


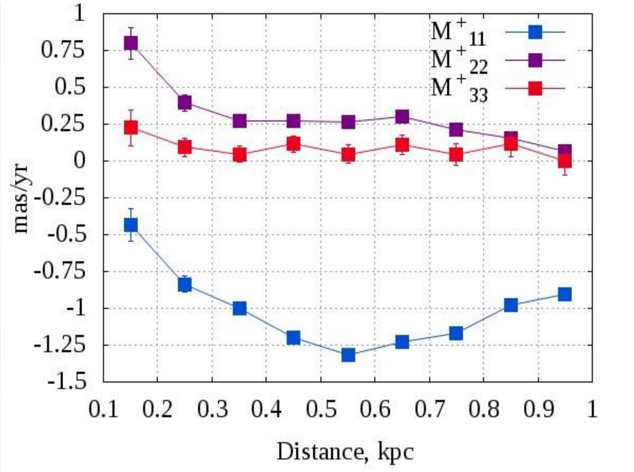












$B = \omega_3 \cdot \eta \qquad A = M^+_{12} \cdot \eta$

$Vrot = (B – A) \cdot \eta \cdot R$ [km/s]

$R = 8.0 \pm 0.2$ km/s Vallée, arxiv:1703.05822, 2017

Vector spherical harmonics

The stellar velocity field can be represented as:

$$\mathbf{U}(l,b) = \sum_{\mathrm{nkp}} \mathrm{t_{nkp}} \mathbf{T}_{\mathrm{nkp}} + \sum_{\mathrm{nkp}} \mathrm{s_{nkp}} \mathbf{S}_{\mathrm{nkp}} + \sum_{\mathrm{nkp}} \mathrm{v_{nkp}} \mathbf{V}_{\mathrm{nkp}}$$

$$\mathbf{T}_{nkp} = T^l_{nkp}\mathbf{e}_l + T^b_{nkp}\mathbf{e}_b = \frac{1}{\sqrt{n(n+1)}} \times \left(\frac{\partial K_{nkp}(l,b)}{\partial b}\mathbf{e}_l - \frac{1}{\cos b}\frac{\partial K_{nkp}(l,b)}{\partial l}\mathbf{e}_b \right)$$

$$\mathbf{S}_{nkp} = S^l_{nkp}\mathbf{e}_l + S^b_{nkp}\mathbf{e}_b = \frac{1}{\sqrt{n(n+1)}} \times \left(\frac{1}{\cos b}\frac{\partial K_{nkp}(l,b)}{\partial l}\mathbf{e}_l + \frac{\partial K_{nkp}(l,b)}{\partial l}\mathbf{e}_b \right)$$

$$\mathbf{V}_{\mathrm{nkp}}(l,b) = K_{\mathrm{nkp}}(l,b)\mathbf{e}_r$$

where

$$K_{nkp}(l,b) = R_{nk} \times \begin{cases} P_{n,0}(b), & k=0, p=1, \\ P_{nk}(b)\sin kl, & k \neq 0, p=0, \\ P_{nk}(b)\cos kl, & k \neq 0, p=1 \end{cases}$$

$$R_{nk} = \sqrt{\frac{2n+1}{4\pi}} \begin{cases} \sqrt{\frac{2(n-k)!}{(n+k)!}}, & k>0, \\ 1, & k=0 \end{cases}$$

(Vityazev et al., 2009)

(Mignard & Klioner, 2012)

The VSH equations in the Galactic coordinate system

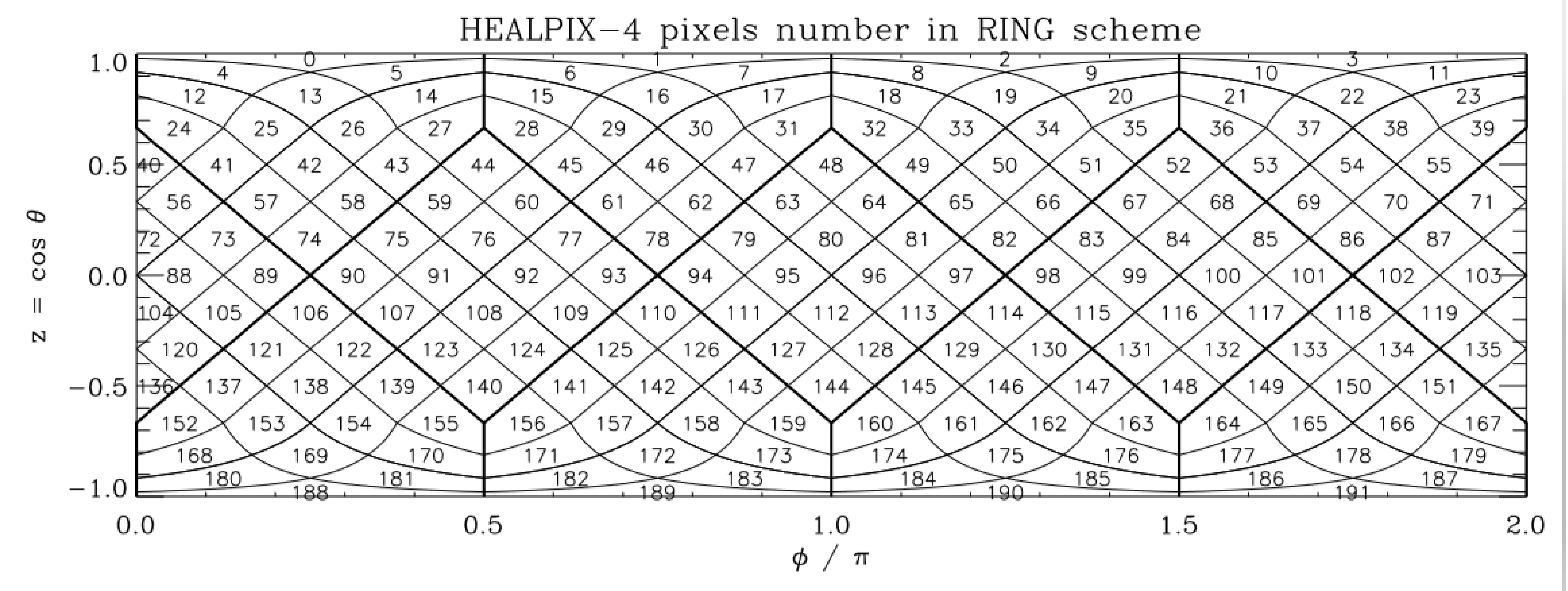
$$\eta\,\mu_l \cos b = \sum_{\mathrm{nkp}} \mathrm{t_{nkp}} \mathrm{T}^{l}_{\mathrm{nkp}}(l,b) + \sum_{\mathrm{nkp}} \mathrm{s_{nkp}}\, \mathrm{S}^{l}_{\mathrm{nkp}}(l,b)$$

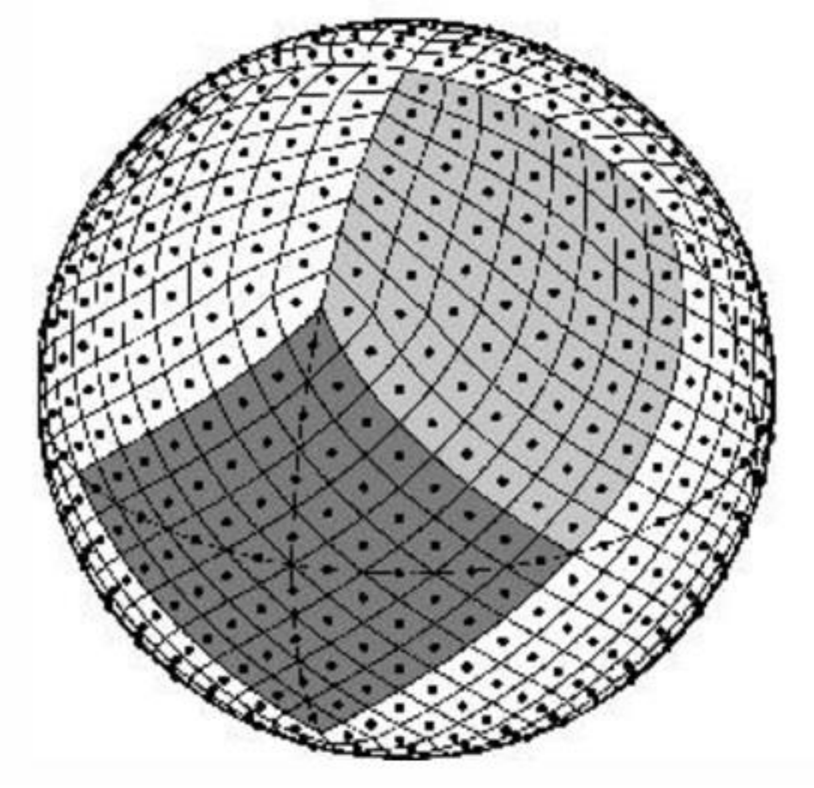
$$\eta\,\mu_b = \sum_{\mathrm{nkp}} \mathrm{t_{nkp}} \mathrm{T}^{b}_{\mathrm{nkp}}(l,b) + \sum_{\mathrm{nkp}} \mathrm{s_{nkp}}\, \mathrm{S}^{b}_{\mathrm{nkp}}(l,b)$$

$$V_r/r = \sum_{\mathrm{nkp}} \mathrm{v_{nkp}} \mathrm{V_{nkp}}(l,b)$$

Preliminary pixelization of data using HEALPix library

(Gorsky et al., ApJ, 622: 759-771, 2005)





1 pixel = 1 star with averaged $\mu_l \cos b$, μ_b and V_r

Npix = 1200

Relations between the OMM parameters and coefficients of decomposition onto vector spherical harmonics

Coefficient	Value
v_{001}	$1.18M^+_{11} + 1.18M^+_{22} + 1.18M^+_{33}$
v_{101}	$-2.05W/\langle r\rangle$
v_{110}	$-2.05V/\langle r\rangle$
v_{111}	$-2.05U/\langle r\rangle$
v_{201}	$-0.53M^+_{11} - 0.53M^+_{22} + 1.06M^+_{33}$
v_{210}	$1.83M^+_{23}$
v_{211}	$1.83M^+_{13}$
v_{220}	$1.83M^+_{12}$
v_{221}	$0.92M^+_{11} - 0.92M^+_{22}$

t_{101}	$2.89\omega_3$
t_{110}	$2.89\omega_2$
t_{111}	$2.89\omega_1$
s_{101}	$-2.89W/\langle r\rangle$
s_{110}	$-2.89V/\langle r\rangle$
s_{111}	$-2.89U/\langle r\rangle$
s_{201}	$-0.65M^+_{11} - 0.65M^+_{22} + 1.29M^+_{33}$
s_{210}	$2.24M^+_{23}$
s_{211}	$2.24M^+_{13}$
s_{220}	$2.24M^+_{12}$
s_{221}	$1.12M^+_{11} - 1.12M^+_{22}$

[Vityazev, Tsvetkov, Astronomy Letters, Vol. 35, № 2, pp. 100-113, 2009]

The VSH method allows to detect **extra-model** systematic components presented in the observed stellar velocity field

harmonic	norm. c.	α	δ	r
s_{310}	$\sqrt{\frac{7}{128\pi}}$	$-\cos l\,(5\sin^2 b-1)$	$\sin b \sin l\,(15\sin^2\ -11)$	
t_{211}	$\sqrt{\frac{5}{8\pi}}$	$-\cos 2b \cos l$	$-\sin b \sin l$	
v_{310}	$\sqrt{\frac{21}{32\pi}}$			$\cos b \sin l\,(5\sin^2 b-1)$

r, pc	± b
400	± 36.9°
500	± 31.0°
600	± 26.5°
700	± 23.1°
800	± 20.5°
900	± 18.4°
1000	± 16.7°

Zonal Vector spherical harmonics

$$Z = \begin{cases} 0 \le l \le 2\pi, \\ b_{\min} \le b \le b_{\max}. \end{cases}$$

Transformation of Galactic latitudes

$$\hat{b} = \arcsin(P \sin b + Q)$$

where

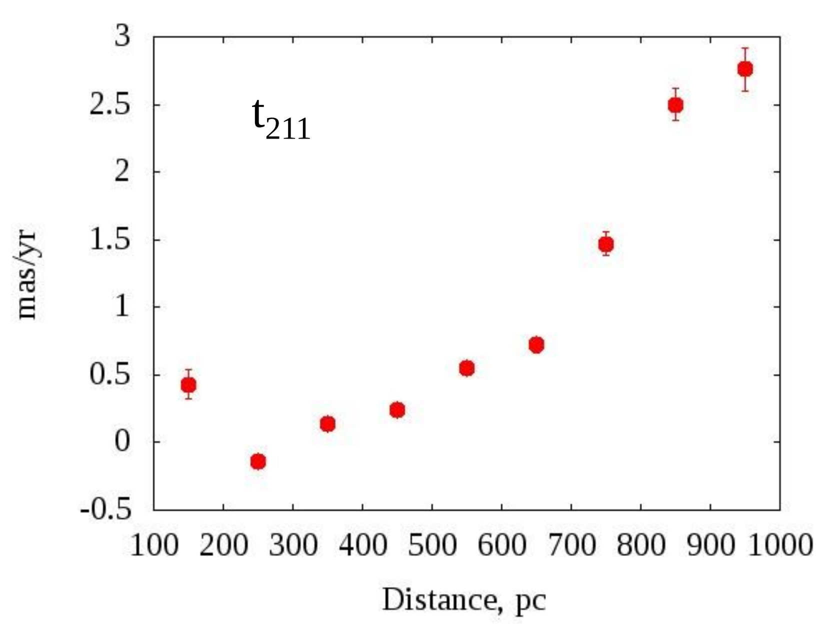
$$P = \frac{2}{s_2 - s_1}, \quad Q = -\frac{s_2 + s_1}{s_2 - s_1},$$

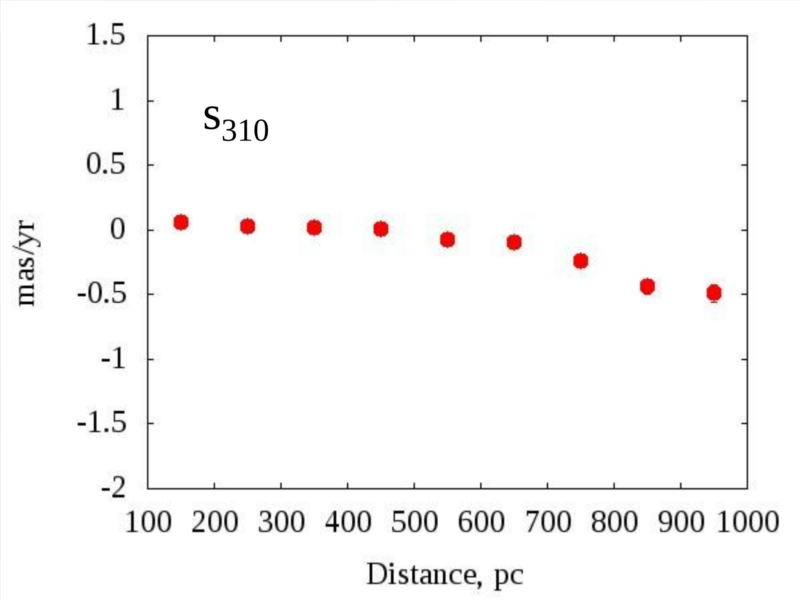
$$s_1 = \sin b_{\min}, \quad s_2 = \sin b_{\max},$$

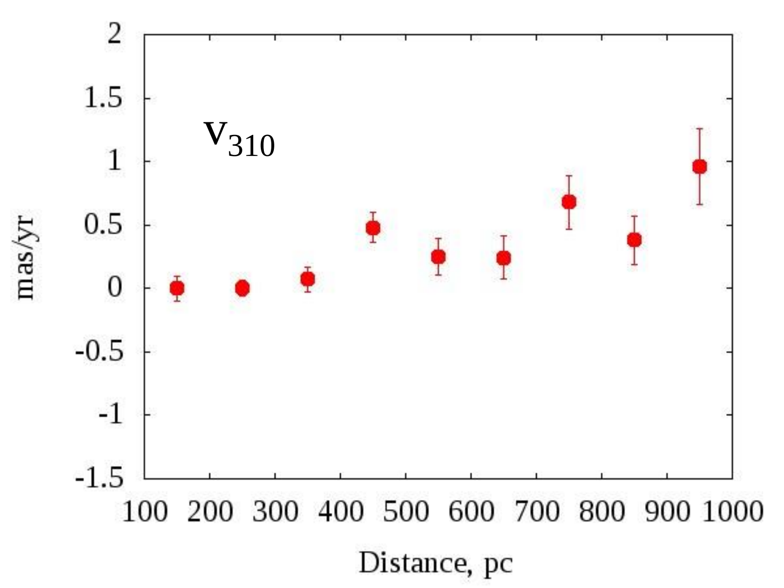
	t_{101}	t_{110}	t_{111}	s_{101}	s_{110}	s_{111}	s_{201}	s_{210}	s_{211}	s_{220}	s_{221}	t_{211}	s_{310}
ω_3	0.397	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
ω_2	0.0	0.557	0.0	0.0	0.0	0.0	0.0	0.0	0.417	0.0	0.0	0.0	0.0
ω_1	0.0	0.0	0.557	0.0	0.0	0.0	0.0	-0.417	0.0	0.0	0.0	0.0	0.0
U/r	0.0	0.0	0.0	0.0	0.0	-0.173	0.0	0.0	0.0	0.0	0.0	0.0	0.0
V/r	0.0	0.0	0.0	0.0	-0.173	0.0	0.0	0.0	0.0	0.0	0.0	0.246	-0.382
W/r	0.0	0.0	0.0	-0.138	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
M^+_{23}	0.0	0.0	-0.130	0.0	0.0	0.0	0.0	0.891	0.0	0.0	0.0	0.0	0.0
M^+_{13}	0.0	0.130	0.0	0.0	0.0	0.0	0.0	0.0	0.891	0.0	0.0	0.0	0.0
M^+_{12}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.541	0.0	0.0	0.0
M^*_{11}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.235	0.0	0.0
M^*_{33}	0.0	0.0	0.0	0.0	0.0	0.0	1.250	0.0	0.0	0.0	0.0	0.0	0.0
T_{211}	0.0	0.0	0.0	0.0	0.083	0.0	0.0	0.0	0.0	0.0	0.0	-1.152	0.990
S_{310}	0.0	0.0	0.0	0.0	0.018	0.0	0.0	0.0	0.0	0.0	0.0	0.103	-0.390

	v_{001}	v_{101}	v_{110}	v_{111}	v_{201}	v_{210}	v_{211}	v_{220}	v_{221}	v_{310}
U/r	0.0	0.0	0.0	-0.197	0.0	0.0	0.0	0.0	0.0	0.0
V/r	0.0	0.0	-0.197	0.0	0.0	0.0	0.0	0.0	0.0	-0.228
W/r	0.0	-0.368	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
M^+_{23}	0.0	0.0	0.0	0.0	0.0	0.885	0.0	0.891	0.0	0.0
M^+_{13}	0.0	0.0	0.0	0.0	0.0	0.0	0.885	0.0	0.0	0.0
M^+_{12}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.610	0.0	0.0
M^+_{11}	1.000	0.0	0.0	0.0	-0.408	0.0	0.0	0.0	0.609	0.0
M^+_{22}	1.000	0.0	0.0	0.0	-0.408	0.0	0.0	0.0	-0.609	0.0
M^+_{33}	1.000	0.0	0.0	0.0	2.994	0.0	0.0	0.0	0.001	0.0
V_{310}	0.0	0.0	-0.200	0.0	0.0	0.0	0.0	0.0	0.0	1.13

Extra-model components T_{211} S_{310} and V_{310}



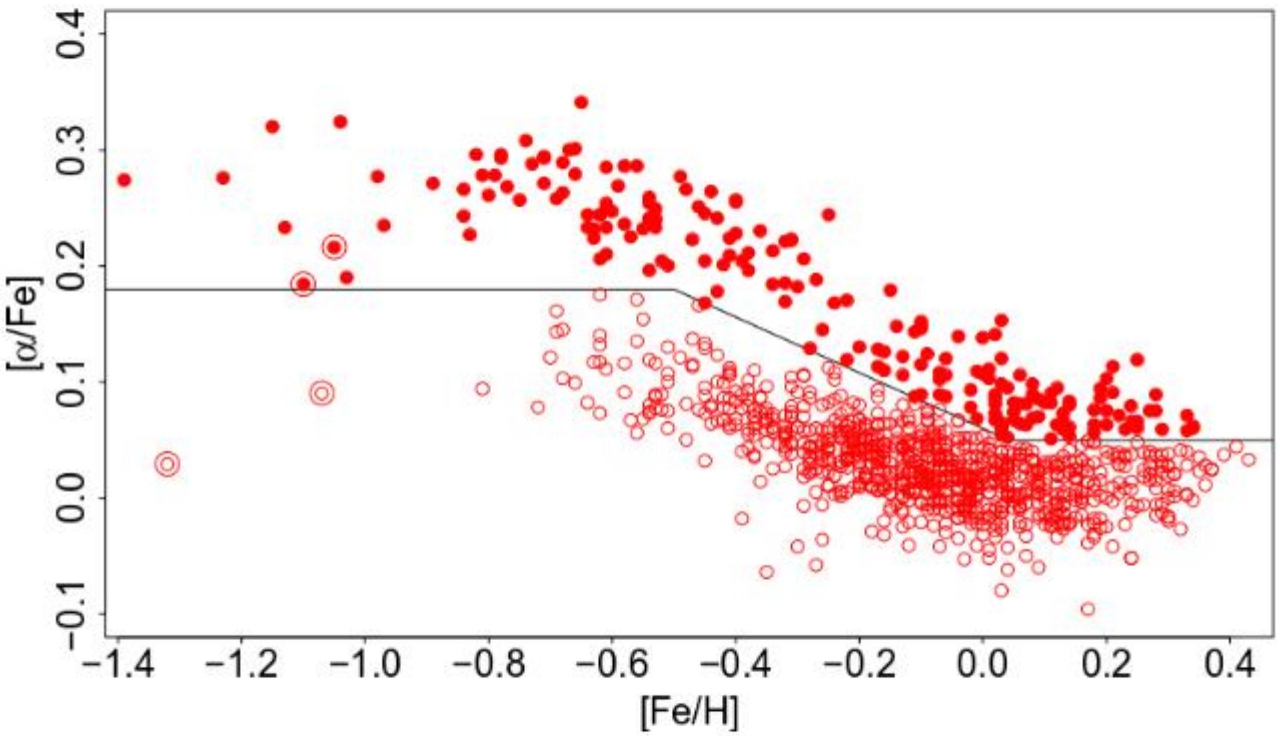




Conclusions

1. По пространственным скоростям Gaia DR2 кинематическим методом Бенсби выделены звезды тонкого диска Галактики
2. Получены кинематические параметры Галактики с использованием модели Огородникова – Милна, а также методом разложения поля скоростей звезд по набору ВСФ
3. Значения кинематических параметров, полученные по трехмерным скоростям и только по собственным движениям Gaia DR2, для данной выборки звезд очень близки.
4. Значения линейной скорости вращения Галактики на расстоянии Солнца, полученные по собственным движениям PMA, систематически меньше значений, полученных по данным Gaia DR2.
5. Подтверждено наличие внемодельных гармоник t_{211}, s_{310}, v_{310}


Thank you
for your attention !
Спасибо
за внимание !



Adebekyan 2012+, Haywood 2013+

From NIST database

Transition	H I, nm	Ca II, nm
3 - 13	866.502	866.214
3 - 15	854.538	854.209
3 - 16	850.249	849.802

Paschen series

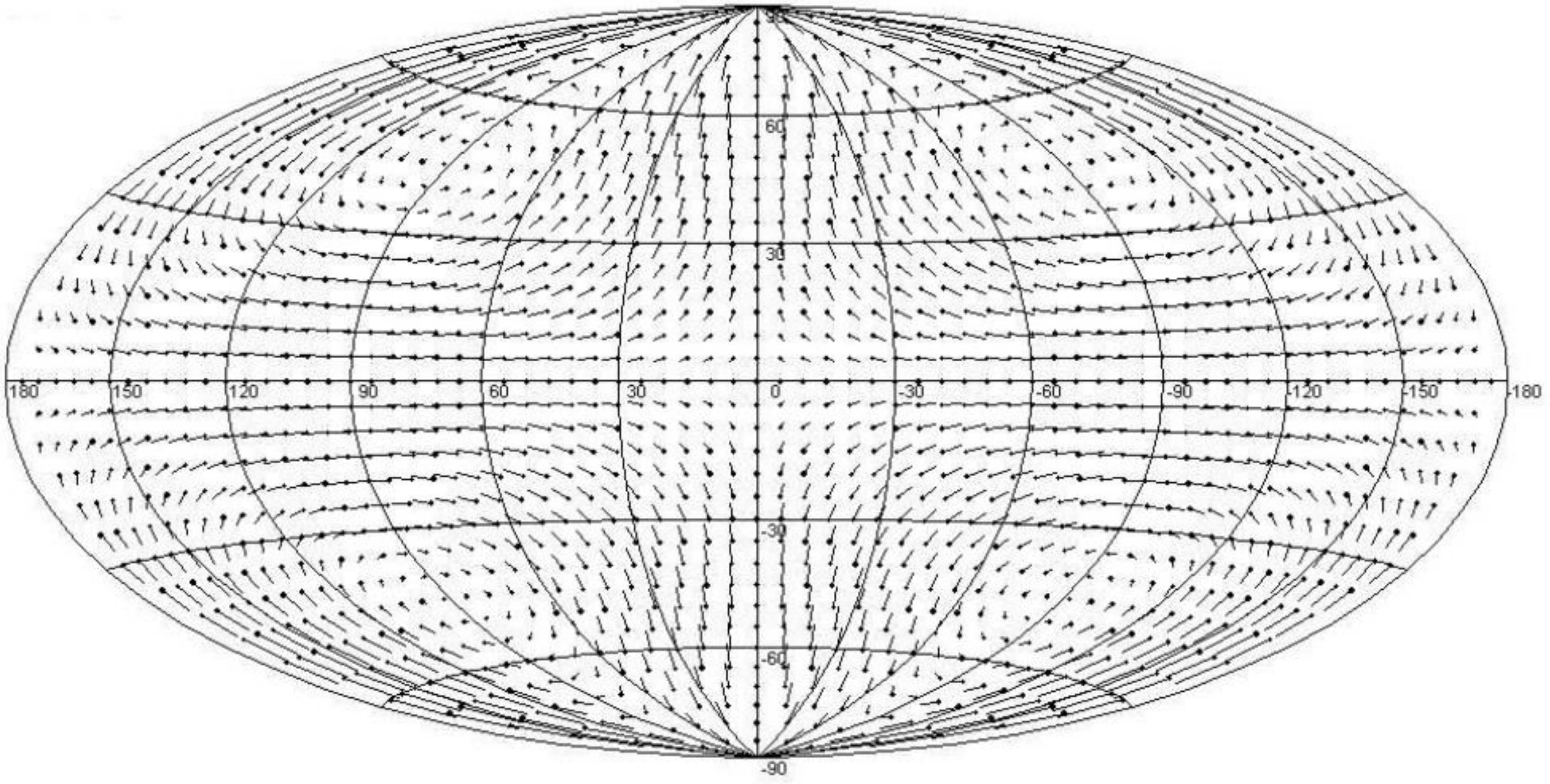
Thoroidal harmonics (Mignard & Klioner, 2012)

Harm.	Mult. coef.	Components $\boldsymbol{e}_\alpha$	$\boldsymbol{e}_\delta$
$\boldsymbol{T}_{10}$	$\frac{1}{2}\sqrt{\frac{3}{2\pi}}$	$\cos\delta$	0
$\boldsymbol{T}_{11}$	$\frac{1}{4}\sqrt{\frac{3}{\pi}}$	$\sin\delta\,(\cos\alpha + \imath\sin\alpha)$	$-\sin\alpha + \imath\cos\alpha$
$\boldsymbol{T}_{20}$	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}$	$\sin 2\delta$	0
$\boldsymbol{T}_{21}$	$\frac{1}{4}\sqrt{\frac{5}{\pi}}$	$-\cos 2\delta\,(\cos\alpha + \imath\sin\alpha)$	$-\sin\delta\,(\sin\alpha - \imath\cos\alpha)$
$\boldsymbol{T}_{22}$	$\frac{1}{8}\sqrt{\frac{5}{\pi}}$	$-\sin 2\delta\,(\cos 2\alpha + \imath\sin 2\alpha)$	$2\cos\delta\,(\sin 2\alpha - \imath\cos 2\alpha)$
$\boldsymbol{T}_{30}$	$\frac{1}{8}\sqrt{\frac{21}{\pi}}$	$\cos\delta\,(5\sin^2\delta - 1)$	0
$\boldsymbol{T}_{31}$	$\frac{1}{16}\sqrt{\frac{7}{\pi}}$	$\sin\delta\,(15\sin^2\delta - 11)\,(\cos\alpha + \imath\sin\alpha)$	$-(5\sin^2\delta - 1)\,(\sin\alpha - \imath\cos\alpha)$
$\boldsymbol{T}_{32}$	$\frac{1}{8}\sqrt{\frac{35}{2\pi}}$	$-\cos\delta\,(3\sin^2\delta - 1)\,(\cos 2\alpha + \imath\sin 2\alpha)$	$\sin 2\delta\,(\sin 2\alpha - \imath\cos 2\alpha)$
$\boldsymbol{T}_{33}$	$\frac{1}{16}\sqrt{\frac{105}{\pi}}$	$\cos^2\delta\sin\delta\,(\cos 3\alpha + \imath\sin 3\alpha)$	$-\cos^2\delta\,(\sin 3\alpha - \imath\cos 3\alpha)$
$\boldsymbol{T}_{40}$	$\frac{3}{16}\sqrt{\frac{5}{\pi}}$	$\sin 2\delta\,(7\sin^2\delta - 3)$	0
$\boldsymbol{T}_{41}$	$\frac{3}{16}\sqrt{\frac{1}{\pi}}$	$(28\sin^4\delta - 27\sin^2\delta + 3)\,(\cos\alpha + \imath\sin\alpha)$	$-\sin\delta\,(7\sin^2\delta - 3)\,(\sin\alpha - \imath\cos\alpha)$
$\boldsymbol{T}_{42}$	$\frac{3}{16}\sqrt{\frac{2}{\pi}}$	$-\sin 2\delta\,(7\sin^2\delta - 4)\,(\cos 2\alpha + \imath\sin 2\alpha)$	$\cos\delta\,(7\sin^2\delta - 1)\,(\sin 2\alpha - \imath\cos 2\alpha)$
$\boldsymbol{T}_{43}$	$\frac{3}{16}\sqrt{\frac{7}{\pi}}$	$\cos^2\delta\,(4\sin^2\delta - 1)\,(\cos 3\alpha + \imath\sin 3\alpha)$	$-3\cos^2\delta\sin\delta\,(\sin 3\alpha - \imath\cos 3\alpha)$
$\boldsymbol{T}_{44}$	$\frac{3}{8}\sqrt{\frac{7}{2\pi}}$	$-\cos^3\delta\sin\delta\,(\cos 4\alpha + \imath\sin 4\alpha)$	$\cos^3\delta\,(\sin 4\alpha - \imath\cos 4\alpha)$

Spheroidal harmonics (Mignard & Klioner, 2012)

Harm.	Mult. coef.	Components e_α	e_δ
S_{10}	$\frac{1}{2}\sqrt{\frac{3}{2\pi}}$	0	$\cos\delta$
S_{11}	$\frac{1}{4}\sqrt{\frac{3}{\pi}}$	$\sin\alpha - \imath\cos\alpha$	$\sin\delta\,(\cos\alpha + \imath\sin\alpha)$
S_{20}	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}$	0	$\sin 2\delta$
S_{21}	$\frac{1}{4}\sqrt{\frac{5}{\pi}}$	$\sin\delta\,(\sin\alpha - \imath\cos\alpha)$	$-\cos 2\delta\,(\cos\alpha + \imath\sin\alpha)$
S_{22}	$\frac{1}{8}\sqrt{\frac{5}{\pi}}$	$-2\cos\delta\,(\sin 2\alpha - \imath\cos 2\alpha)$	$-\sin 2\delta\,(\cos 2\alpha + \imath\sin 2\alpha)$
S_{30}	$\frac{1}{8}\sqrt{\frac{21}{\pi}}$	0	$\cos\delta\,(5\sin^2\delta - 1)$
S_{31}	$\frac{1}{16}\sqrt{\frac{7}{\pi}}$	$(5\sin^2\delta - 1)\,(\sin\alpha - \imath\cos\alpha)$	$\sin\delta\,(15\sin^2\delta - 11)\,(\cos\alpha + \imath\sin\alpha)$
S_{32}	$\frac{1}{8}\sqrt{\frac{35}{2\pi}}$	$-\sin 2\delta\,(\sin 2\alpha - \imath\cos 2\alpha)$	$-\cos\delta\,(3\sin^2\delta - 1)\,(\cos 2\alpha + \imath\sin 2\alpha)$
S_{33}	$\frac{1}{16}\sqrt{\frac{105}{\pi}}$	$\cos^2\delta\,(\sin 3\alpha - \imath\cos 3\alpha)$	$\cos^2\delta\sin\delta\,(\cos 3\alpha + \imath\sin 3\alpha)$
S_{40}	$\frac{3}{16}\sqrt{\frac{5}{\pi}}$	0	$\sin 2\delta\,(7\sin^2\delta - 3)$
S_{41}	$\frac{3}{16}\sqrt{\frac{1}{\pi}}$	$\sin\delta\,(7\sin^2\delta - 3)\,(\sin\alpha - \imath\cos\alpha)$	$(28\sin^4\delta - 27\sin^2\delta + 3)\,(\cos\alpha + \imath\sin\alpha)$
S_{42}	$\frac{3}{16}\sqrt{\frac{2}{\pi}}$	$-\cos\delta\,(7\sin^2\delta - 1)\,(\sin 2\alpha - \imath\cos 2\alpha)$	$-\sin 2\delta\,(7\sin^2\delta - 4)\,(\cos 2\alpha + \imath\sin 2\alpha)$
S_{43}	$\frac{3}{16}\sqrt{\frac{7}{\pi}}$	$3\cos^2\delta\,\sin\delta\,(\sin 3\alpha - \imath\cos 3\alpha)$	$\cos^2\delta\,(4\sin^2\delta - 1)\,(\cos 3\alpha + \imath\sin 3\alpha)$
S_{44}	$\frac{3}{8}\sqrt{\frac{7}{2\pi}}$	$-\cos^3\delta\,(\sin 4\alpha - \imath\cos 4\alpha)$	$-\cos^3\delta\,\sin\delta\,(\cos 4\alpha + \imath\sin 4\alpha)$

T_{210} (Витязев & Цветков, 2009)



S210 (Витязев & Цветков, 2009)