Continuous Wavelet Transform in Quantum Field Theory: An application to the Casimir effect

Mikhail V. Altaisky[1,2] Natalia E.Kaputkina[3]

[1] Space Research Institute RAS, Profsoyuznaya 84/32, Moscow, 117997, Russia
[2] Joint Institute for Nuclear Research, Joliot Curie 6, Dubna, 141980, Russia
[3] National University of Science and Technology "MISiS", Leninsky prospect 4, Moscow, 119049, Russia
altaisky@mx.iki.rssi.ru, nataly@misis.ru

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Abstract

The Casimir force $F = -\frac{\pi^2 \hbar c}{240 l^4}$, which attracts to each other two perfectly conducting parallel plates separated by a distance a in vacuum, is one of the blueprints of the reality of vacuum fluctuations. The physical basis for calculation of the Casimir force is quantum electrodynamics defined on the space of point-dependent, possibly singular, functions $\phi(x)$, the distributions. On this space the Green functions are defined. We have shown that it is possible to use region-dependent functions $\phi_a(x)$, where a is the size of the region centered around the space-time point x, constructed by means of the continuous wavelet transform, to get finite Green functions by construction. The singular limit of point-dependent fields is restored by integration over all scales a. We suggest that real measurement of the Casimir force between the parallel plates separated by the distance l is also dependent on the plate displacement $l \to l - \delta$, i.e., $F = F(l, \delta/l)$. The particular form of this dependence is derived.

References:

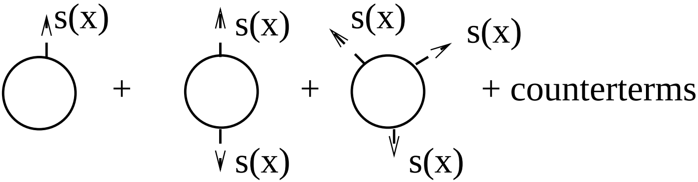
This talk is based on

1. M.V.Altaisky. *Phys. Rev. D* 81(2010) 125003
2. M.V.Altaisky and N.E.Kaputkina. *JETP Lett.* 94(2011)341
3. A.M.Frassino and O.Panella. *Phys. Rev. D* 85 (2012) 045030
4. M.V.Altaisky and N.E.Kaputkina. *Phys. Rev. D* 88(2013)025015

Casimir force results from the difference of the vacuum energy of the two different configurations: the rectangular volume bounded by two parallel conducting walls, and that not bounded by walls.
Due to the Casimir effect two parallel conducting planes in vacuum attract each other with a force distance between the plates *a* [Cas48]:

$$F_C = -\frac{\hbar c \pi^2}{240 a^4} = -\frac{1.3 \cdot 10^{-27} N m^2}{a^4}$$

In QED the Casimir effect is described by the loop diagrams



where external field s(x) desribes the boundary conditions

QED as a field theory depending on scale

In quantum field theory the state of a quantum field $|\phi\rangle$ is described by the function $phi(x) = \langle x|\phi\rangle$. The calculation of Feynman diagrams is performed in Fourier representation

$$\langle x|\phi\rangle = \int \langle x|p\rangle dp \langle p|\phi\rangle$$

It is also possible to use other locally compact groups to represent quantum fields

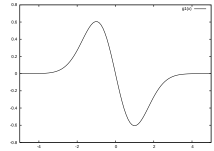
$$|\phi\rangle = \frac{1}{C_g} \int_G U(\nu)|g\rangle d\mu(\nu) \langle g|U^*(\nu)|\phi\rangle$$

A particular case of the affine group $G : x' = ax + b$ is known as

Wavelet transform

$$\phi_a(x) \equiv \langle x, a; g|\phi\rangle$$

$$\phi_a(x) = \int \frac{1}{a}\bar{g}\left(\frac{x-b}{a}\right)\phi(x)dx$$



Wavelet-based regularization

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$$\phi(x) \to \phi_a(x) \Rightarrow W_W[J_a(x)] = \int e^{-\tilde{S}[\phi_a(x)] + \int J_a(x)\phi_a(x)\frac{dadx}{a}} \mathcal{D}\phi_a(x)$$

Scale-dependent fields

$\phi(x) \to \phi_\delta(x)$, where x is position, δ is resolution

- each field $\tilde{\phi}(k)$ will be substituted by the scale component $\tilde{\phi}_\delta(k) = \overline{\tilde{g}(\delta k)}\tilde{\phi}(k)$.
- each integration in momentum variable will be accompanied by integration in corresponding scale variable:

$$\frac{d^d k}{(2\pi)^d} \to \frac{d^d k}{(2\pi)^d} \frac{d\delta}{\delta}.$$

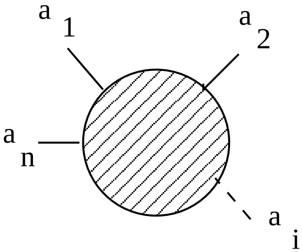
- each vertex is substituted by its wavelet transform.

Green functions

$$\langle \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle_c = \left.\frac{\delta^n \ln W_W[J_a]}{\delta J_{a_1}(x_1)\ldots\delta J_{a_n}(x_n)}\right|_{J=0}.$$

$$G_0^{(2)}(a_1,a_2,p) = \frac{\tilde g(a_1 p)\tilde g(-a_2 p)}{p^2+m^2}.$$

The integration over the internal scale variables $\frac{da_i}{a_i}$ in each loop is performed from the minimal scale of all extermal lines



$A = \min(a_1,\ldots,a_n)$ up to infinity.

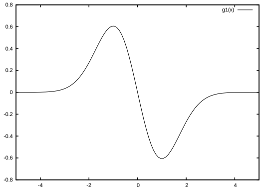
Effective cutoff

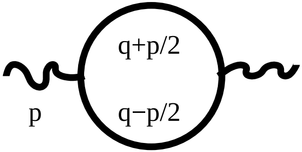
The aperture function

$$g(x) = -xe^{-x^2/2},$$

up to appropriate rescaling, after the integration in internal lines leads to the cutoff function

$$F_A(p) = e^{-4\delta^2 p^2}.$$





Regularization

Both integrals above, for the discrete and continuous spectra, are evidently infinite, but their difference

$$\mathcal{E} = \frac{E_Q - E_0}{L_x L_y},$$

– the Casimir energy, can be regularized if the integrands are multiplied by some cutoff function $f(k)$, such that

$$f(0) = 1, \text{ and } f\left(k \gg \frac{1}{a_0}\right) \to 0,$$

where $1/a_0$ is the inverse size of atom [IZ80]. This choice accounts for the fact, that the walls are made of real atoms.

The dimensional regularization is often used as well.

Regularized result gives the known values of the Casimir energy and the Casimir force:

$$E(a) = -\frac{\hbar c \pi^2}{720 a^3}, \quad F(a) = -\frac{\hbar c \pi^2}{240 a^4}$$

Scale-dependent Casimir energy

After the choice $f(k) = e^{-4\delta^2 k^2}$ the regularized Casimir energy is:

$$\begin{aligned}
\mathcal{E} &= \hbar c \frac{\pi^2}{4a^3}\left[\frac{1}{2}F(0) + F(1) + \ldots - \int_0^\infty F(n)dn\right], \\
F(n) &= \int_0^\infty du\sqrt{u+n^2}\exp\left(-4\pi^2\frac{\delta^2}{a^2}(u+n^2)\right) \\
&= \frac{\sqrt{\pi}}{2\left(\frac{2\pi\delta}{a}\right)^3}\left[1-\mathrm{erf}\left(\frac{2\pi\delta}{a}n\right) + 4\sqrt{\pi}\exp\left(-\left(\frac{2\pi\delta}{a}\right)^2 n^2\right)\frac{\delta}{a}n\right].
\end{aligned}$$

The difference between the sum and the integral above is evaluated by Euler-Maclaurin formula

$$\frac{1}{2}F(0)+F(1)+\ldots-\int_0^\infty F(n)dn = -\frac{1}{2!}B_2F'(0)-\frac{1}{4!}B_4F'''(0)-\ldots,$$

where B_n are the Bernoulli numbers.

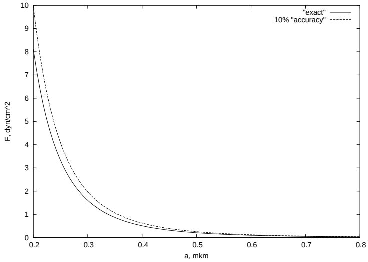
This gives the Casimir energy

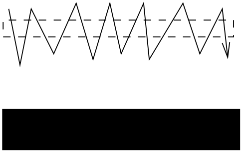
$$\mathcal{E}(a,\delta) = -\frac{\hbar c\pi^2}{720a^3}\left[1+\frac{2}{7}\left(\frac{2\pi\delta}{a}\right)^2 + \frac{3}{28}\left(\frac{2\pi\delta}{a}\right)^4+\ldots\right],$$

and the Casimir force

$$\mathcal{F}(a,\delta) = -\frac{\hbar c\pi^2}{240a^4}\left[1+\frac{10}{21}\left(\frac{2\pi\delta}{a}\right)^2 + \frac{1}{4}\left(\frac{2\pi\delta}{a}\right)^4+\ldots\right],$$

Deviation of Casimir force between two plates of unit area in vacuum. The solid line denotes the "exact" Casimir force ($\delta=0$), the dashed line denotes the scale-dependent Casimir force with $\delta/a=0.1$





$z(t) = z_0 + \delta \cos wt$

What is the dependence of frequency shift on δ ?

$$F(a,\delta) = -\frac{E(a_2) - E(a_1)}{\delta}, \delta = a_2 - a_1$$

Standard theory

$$F(a,\delta) = -\frac{\hbar c \pi^2}{240 a^4}\left[1 - 4\frac{\delta}{a} + 10\left(\frac{\delta}{a}\right)^2 - \ldots\right]$$

Wavelet-regularized theory

$$\mathcal{F}(a,\delta) = -\frac{\hbar c \pi^2}{240 a^4}\left[1 + \frac{10}{21}\left(\frac{2\pi\delta}{a}\right)^2 + \frac{1}{4}\left(\frac{2\pi\delta}{a}\right)^4 + \ldots\right],$$

THANK YOU FOR YOUR ATTENTION !!!

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