Gravitational four-fermion interaction and dynamics of the early Universe

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According to common beliefs, present expansion of Universe is result of Big Bang. Quite popular idea is that this expansion had been preceded by compression with subsequent Big Bounce.

We demonstrate that contrary to popular beliefs

gravitational four-fermion interaction

does not result in Big Bounce.

1. Interaction of fermions with gravity results, due to torsion, in four-fermion interaction (Kibble; Rodichev; Perez, Rovelli; Freidel, Minic, Takeuchi)

$$S_{ff} = \frac{3}{2}\pi\, G\frac{\gamma^2}{\gamma^2+1}\int d^4x\, e\,\Big[\eta_{IJ}A^I A^J - \frac{\alpha}{\gamma}\,\eta_{IJ}(V^I A^J + A^I V^J) - \alpha^2\,\eta_{IJ}V^I V^J\Big].$$

This interaction, proportional to Newton constant G and to particle number density squared n^2, gets essential on the Planck scale only.

We need energy-momentum tensor (EMT) $T_{\mu\nu}$ generated by this action. Metric tensor enters this action via $\sqrt{-g}$ only (!), thus

$$\frac{1}{2}\sqrt{-g}\,T_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}} S_{ff}\,.$$

With identity

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\,g_{\mu\nu}\,,$$

we arrive at the following expression for EMT:

$$T_{\mu\nu} = -\frac{3\pi}{2}G\,\frac{\gamma^2}{\gamma^2+1}\,g_{\mu\nu}[\eta_{IJ}A^I A^J + \frac{\alpha}{\gamma}\,\eta_{IJ}\,(V^I A^J + A^I V^J) - \alpha^2\,\eta_{IJ} V^I V^J].$$

Nonvanishing components of this expression, written in locally inertial frame, are energy density $T_{00} = \rho_{ff}$ and pressure $T_{11} = T_{22} = T_{33} = p_{ff}$.
Thus, equation of state (EOS) is

$$\rho_{ff} = -\,p_{ff} = -\,\frac{\pi}{48}\,G\,\frac{\gamma^2}{\gamma^2+1}\,n^2\,[(3 - 11\,\zeta) - \alpha^2(60 - 28\,\zeta)]\,.$$

Here n is total density of fermions and antifermions, $\zeta = <\sigma_a \sigma_b>$ is average value of product of corresponding σ-matrices, presumably universal for any a and b. With large number of sorts of fermions and antifermions, we neglect here for numerical reasons contributions of exchange and annihilation contributions, and the fact that, if σ_a and σ_b refer to same particle, $<\sigma_a \sigma_b> = 3$.
After averaging over momenta orientations, P-odd contributions of VA to ρ_{ff} and p vanish.

At last, there are no reasons to expect that $\zeta = < \sigma_a \sigma_b >$ can survive under the extreme conditions. Thus, EOS simplifies to

$$\rho_{ff} = -p_{ff} = -\frac{\pi}{16} G \frac{\gamma^2}{\gamma^2+1} n^2 (1 - 20\alpha^2) .$$

From now on, energy density is rewritten as

$$\rho_{ff} = G n^2 \varepsilon,$$

with

$$\varepsilon = -\frac{\pi}{16} \frac{\gamma^2}{\gamma^2+1} (1 - 20\alpha^2) .$$

2. Common matter on the Planck scale is ultrarelativistic. Its energy density is

$$\rho = \nu n^{4/3},$$

ν is numerical factor, $n^{1/3}$ is typical energy per particle.
Another factor n is total density of ultrarelativistic particles and antiparticles, fermions and bosons, contributing to this density. This factor exceeds fermion density n entering above four-fermion expressions. This difference is absorbed here by factor ν. In corresponding EMT, due to isotropy,

$$T_{0m} = T_{m0} = 0, \qquad T_{11} = T_{22} = T_{33}.$$

Since $T^{\mu}_{\mu} = 0$,

$$T^{\mu}_{\nu} = \rho \, \mathrm{diag}(1, -1/3, -1/3, -1/3),$$

$$\text{or} \quad T_{\mu\nu} = \rho \, \mathrm{diag}(1,\ 1/3,\ 1/3,\ 1/3)\,; \quad \text{thus } p = \rho/3.$$

3. We assume that, even when EOS reduces to this one, Universe is homogeneous and isotropic, and is described by Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - a(t)^2[\,dr^2 + f(r)(d\theta^2 + \sin^2\theta\, d\phi^2)];$$

$f(r)$ depends on topology of Universe as a whole:

$$f(r) = r^2,\ \sin^2 r,\ \sinh^2 r$$

for spatial flat, closed, and open Universe, respectively.

Einstein equations for FLRW metric reduce now to

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_{ff} + \rho\right), \qquad \frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left(\rho_{ff} - \rho\right);$$

here k equals 0, 1, and -1 for spatial flat, closed, and open Universe, respectively. Observational data strongly favor the idea that our Universe is spatial flat, i.e. that $k = 0$. Above equations are supplemented by covariant continuity equation:

$$\dot{\rho}_{ff} + \dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0.$$

4. Solution of FLRW equations. With substitution

$$a(t) = a_0 \exp(f(t)) \,, \tag{1}$$

continuity equation is satisfied identically. Two other equations result in

$$\frac{8\pi G}{3}(\rho_{ff} + \rho) = \dot{f}^{\,2}, \quad \frac{8\pi G}{3}\rho = -\frac{1}{2}\ddot{f} \,.$$

Differentiating first of them and combining with second, we arrive at

$$f = -\frac{3}{4\nu} G \,\varepsilon\, n^{2/3} - \frac{1}{3}\ln n \,, \quad \varepsilon = -\frac{\pi}{16}\frac{\gamma^2}{\gamma^2 + 1}(1 - 20\alpha^2) \,.$$

and thus

$$a(t) = a_0 \exp(f(t)) \sim n^{-1/3} \exp\{-\frac{3}{4\,\nu} G\, \varepsilon\, n^{2/3}\}.$$

1) $\varepsilon > 0$. In compression, at first, both factors shrink to zero. Rewrite previous equations as

$$\dot{a} = -\sqrt{\frac{8\pi G}{3}}\, a\, \sqrt{\rho_{ff} + \rho}, \qquad \ddot{a} = \frac{8\pi G}{3}\, a\, (\rho_{ff} - \rho).$$

At first, when $\rho_{ff} \ll \rho$, both $\dot{a}$ and $\ddot{a}$ are negative, Universe shrinks with acceleration. At $\rho_{ff} = \rho$ acceleration $\ddot{a}$ changes sign, while $\dot{a}$ remains negative, and compression of Universe decelerates.

Rather tedious calculations demonstrate that
1. it takes finite time for a to shrink to zero,
2. $\dot{a}$ and $\ddot{a}$ also vanish at the same moment.

Repulsive GFFI does not stop the collapse, but only reduces its rate.
The asymptotic behavior of $a(t)$ is

$$a(t) \sim (t_1 - t) \exp\left(-\frac{9\,\varepsilon^2 G}{128\pi\nu^3}\frac{1}{(t_1 - t)^2}\right),$$

t_1 is the moment of the collapse for $\varepsilon > 0$.

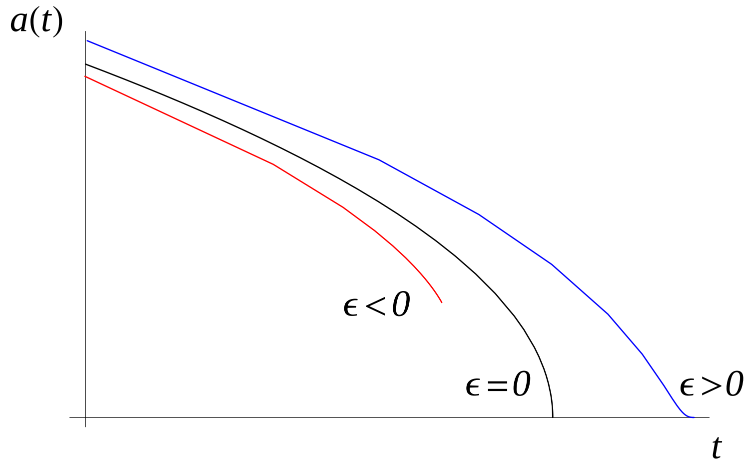

Figure 1: Dependence of scale factor on time for compression

For negative ε, situation is different. Here right-hand side of relation

$$a(t) \sim \frac{1}{\sqrt{|\xi(t)|}}\, e^{\frac{3}{4}|\xi(t)|}, \qquad \xi(t) = \frac{\rho_{ff}}{\rho} = \frac{G\,\varepsilon}{\nu}\, n^{2/3}.$$

has minimum at $|\xi_m| = 2/3,$ i.e., $a(t)$ cannot decrease further. However, the compression rate $\dot{a}$ at this point does not vanish and remains finite. In a sense, the situation here resembles that in the standard cosmology with ultrarelativistic particles: therein $a(t) \sim \sqrt{t_0 - t} \longrightarrow 0$ for $t \longrightarrow t_0$ (t_0 is the moment of the collapse in this case), though at this point $\dot{a}$ is not finite, but tends to infinity.

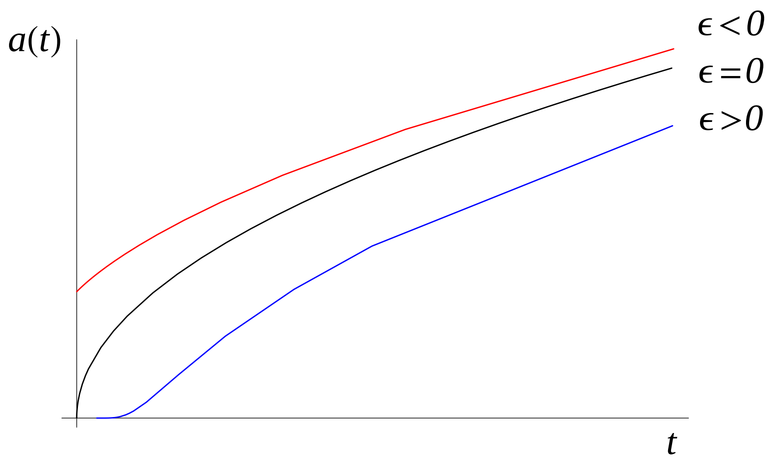

Figure 2: Dependence of scale factor on time for expansion

Gravitational four-fermion interaction

does not result in Big Bounce!